General Relativity and Cosmology : JAP 2000

Lecture notes: Biman B. Nath , Raman Research Institute

biman@rri.ernet.in

1 Introduction to general relativity

There is a common misconception that the special theory of relativity (SR) cannot handle accelerations. This stems from the fact that SR holds in inertial frames and an accelerated frame is noninertial. However, SR can handle accelerations. All one has to do is to move into an instantaneously inertial frame. The space and time intervals in SR do not depend on accelerations and only on the relative velocity. If we want to find out what happens inside an accelerated frame, as seen by an inertial observer in the laboratory, all we have to do is to work out the physics in a given inertial frame, move to the instantaneously comoving frame to see what the observer in the accelerated frame would see instantaneously. This is of course what is done in Newtonian physics too when one discusses the physics in an accelerated frame, which basically leads to the concept of coriolis force etc.

Now what one wants to know is how these different comoving frames fit together as a function of space and time, and this is where general theory of relativity (GR) comes in. In addition, general theory relativity also embodies gravity.

One cornerstone of GR is the observation that a frame under free fall in a graviational field, acts as an inertial frame (locally! – we will soon see what 'locally' means here).

There is an interesting episode in Jules Verne's science fiction '*A trip from the Earth to the Moon*'. The story has it that a projectile containing three men and several animals is fired from a massive cannon pointing skyward. As the projectile (which is unpowered) goes toward Moon, its passengers walk normally inside the projectile on the end nearer Earth. As the projectile continues, they find themselves pressed less and less against the floor, and then reach a point when they float freely, where the gravitational effect of Earth and Moon cancel. Later, they walk around again, but this time on the end nearer Moon. But a dog, named Satellite, dies due to some accident and the passengers throw its remains through a door and find that the body is floating outside the window during the entire trip.

Now, this leads to a paradox. And this is crucial to relativity. Verne thought that the dog should remain close to the ship since both ship and the dog independently follow the same path through space. But why don't the passengers float inside the spaceship if the dog floats outside the ship during the entire trip? If we sawed the spaceship into half, so that the passengers would be outside, would they float freely?

Well, we do know now that the passengers would float freely , at least from the experience of the astronauts. This led Einstein to think in the following terms. From the time of Galileo and Newton one has always thought of gravity as a force. When we throw a ball up in the air, it comes down in a parabola, and we say that this is due to something called 'gravity'. Einstein thought that if we see things in a freely falling frame, if the floor on which we are standing, were to fall freely, then the ball would simply go in a straight line and hit the roof. The freely falling frame is then a more natural frame to see things from. We get the strange effects, like balls moving in a parabola, because we are not seeing things in the right frame. So, according to Einstein, it is as if gravity did not exist, and we could explain things if we knew how to switch from the inertial, freely-falling frames to other 'unnatural' frames (like one standing on the surface of the Earth).

Of course, if the freely falling frame were too big, then it would not remain completely inertial. Imagine Einstein's train falling toward Earth, and we would see that there would be relative acceleration between particles at the two end of the train, simply because Earth's gravitational field is not homogeneous. This is what we meant by saying that 'locally' a freely falling frame is an inertial frame.

But let us look at this more carefully. We just saw that because of tidal effects, the separation of particles would change. This gives us a hint that geometry is somewhat is linked to gravity, or gravity manifests itself in the geometry of spacetime. Although Einstein thought of the geometry of spacetime, we can illustrate the idea in terms of two-dimensional geometry on the surface of sphere.

Recall that in Newtonian gravity, one has the problem of identifying two kinds of masses of an object – the interal and graviational mass. Experiments (beginning with Galileo) show that these two masses are same for all practical purposes. This equality needs to be added to the Newtonian system of equations; it is an extra thing.

Now consider two travellers setting out from the equator on Earth going towards North pole. Suppose their initial separation is recorded. After travelling some distance, one would find that the distance between them has shrunk, although they were both going north and so were going parallel to each other. Well, if we did not know about the curvature of the surface of Earth, we would have ascribed this strange behaviour – the acceleration toward each other– in terms of mysterious force. We would also notice something interesting. We would find that this 'acceleration' is the same if you walk, or ride a bicycle or drive a massive car. We would conclude that this mysterious force acts on all bodies in the same way, no matter what they are made of or how massive they are.

It was Einstein's idea to think of gravity in terms of the underlying 'geometry' of spacetime. What is the cause of the curvature of spacetime? We will see that it is the energy and momentum density that causes spacetime to become curved. But it is not a one-way effect though. Spacetime also acts, in Einstein's equation, on energy and momentum telling a particle how to move. But that is going abit too far ahead. To begin with we will first learn to handle vectors and tensors in SR, in flat Minkowskian space.

2 Vectors in Special Relativity

A typical vector would be a displacement vector, which points from one event to another and its components are the coordinate differences. We will write it as,

$$\Delta\vec{x} \quad \underset{O}{\longrightarrow} \quad (\Delta t, \Delta x, \Delta y, \Delta z) \,. \tag{1}$$

By this we mean that the vector $\Delta\vec{x}$ has these components in the frame O. We will also write this as,

$$\Delta\vec{x} \quad \underset{O}{\longrightarrow} \quad \{\Delta x^{\alpha}\} \,. \tag{2}$$

The vectors transform according to the Lorentz transformation rules. We know from SR (special relativity) that,

$$\Delta x^{\overline{o}} = \frac{\Delta x^0}{\sqrt{(1-v^2)}} - \frac{v\Delta x^1}{\sqrt{(1-v^2)}} \,, \tag{3}$$

etc. That is,

$$\Delta x^{\overline{o}} = \Lambda^{\overline{o}}_{\beta}\Delta x^{\beta} \,, \tag{4}$$

where

$$\Lambda^{\overline{o}}_{0} = \frac{1}{\sqrt{(1-v^2)}}, \Lambda^{\overline{o}}_{1} = \frac{-v}{\sqrt{(1-v^2)}}, \Lambda^{\overline{o}}_{2} = \Lambda^{\overline{o}}_{3} = 0 \,. \tag{5}$$

We will write in general (for all α)

$$\Delta x^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}_{\beta}\Delta x^{\beta} \,, \tag{6}$$

where $\Lambda^{\overline{\alpha}}_{\beta}$ are a collection of 16 numbers, that is the Lorentz transformation matrix.

We have in the last equation introduced the summation convention— whenever an index appears as a subscript and a superscript in an expression, a summation is implied over all the values that the index can take. For indices, Greek indices will take all values whereas Latin indices will only take the spatial values (1,2,3). In other words,

$$\Delta x^{\overline{\alpha}} = \Lambda^{\overline{\alpha}}_{0}\Delta x^{0} + \Lambda^{\overline{\alpha}}_{i}\Delta x^{i} \,. \tag{7}$$

2.1 Basis vector

Basis vectors defined in frame O as,

$$\vec{e}_0 \underset{O}{\longrightarrow} (1,0,0,0), \qquad \vec{e}_1 \underset{O}{\longrightarrow} (0,1,0,0), \qquad \vec{e}_2 \underset{O}{\longrightarrow} (0,0,1,0), \qquad \vec{e}_3 \underset{O}{\longrightarrow} (0,0,0,1) \,. \tag{8}$$

Essentially, $(\vec{e}_{\alpha})^{\beta} = \delta^{\beta}_{\alpha}$, which is the Kronecker delta. Any vector can be expressed in terms of these basis vectors, as,

$$\vec{A} = A^{\alpha}\vec{e}_{\alpha} \,. \tag{9}$$

Let us see how the basis vectors transform.

$$\vec{A} = A^{\alpha}\vec{e}_{\alpha} = A^{\overline{\alpha}}\vec{e}_{\overline{\alpha}} = \Lambda^{\overline{\alpha}}_{\beta}A^{\beta}\vec{e}_{\overline{\alpha}} \tag{10}$$

Now, the order of numbers do not matter and also we will change the indices $\beta \to \alpha$ and $\overline{\alpha} \to \overline{\beta}$ since they are dummy indices anyway.

$$A^{\alpha}\Lambda^{\overline{\beta}}_{\alpha}\vec{e}_{\overline{\beta}} = A^{\alpha}\vec{e}_{\alpha} \,. \tag{11}$$

Since this expression is true for an arbitrary vector A^α, we must have (for all values of α)

$$\vec{e}_\alpha = \Lambda^{\overline{\beta}}_{\alpha} \vec{e}_{\overline{\beta}}. \tag{12}$$

Compare this with the transformation rule for vectors,

$$A^{\overline{\beta}} = \Lambda^{\overline{\beta}}_{\alpha} A^\alpha . \tag{13}$$

2.2 An example of a vector

Let us look at the four velocity as an example of a vector. For a uniformly moving particle, the four velocity is defined as an vector which is tangent to the world-line, and of length that is one unit of time in the particle's frame. In other words, it is basically the basis vector $\vec{e}_0$ in its rest frame. For an accelerated particle too, we can use the same definition, provided we do this in a *momentarily comoving reference frame* (MCRF) at a given point in the world-line. That is, $\vec{U} = \vec{e}_{\overline{0}}$ in the MCRF.

Suppose, a particle is moving with velocity $\vec{v}$ in the x-direction of frame O. What are its components in O. In its rest frame, we have, $\vec{U} = \vec{e}_{\overline{0}}$. So, we have in O,

$$U^\alpha = (\vec{e}_{\overline{0}})^\alpha = \Lambda^{\alpha}_{\overline{\beta}} (\vec{e}_{\overline{0}})^\beta = \Lambda^{\alpha}_{\overline{0}} . \tag{14}$$

This means that $U^0 = \frac{1}{\sqrt{(1-v^2)}}, U^1 = \frac{v}{\sqrt{(1-v^2)}}, U^2 = U^3 = 0$. For small v, the spatial components are $v,0,0)$ which justifies the definition of the four velocity.

Suppose a particle makes an instantaneous displacement $d\vec{x}$, whose components in the frame O are (dt,dx,dy,dz). The magnitude of this displacement is $-dt^2 + dx^2 + dy^2 + dz^2$ which is essentially the spacetime interval $ds^2 = d\vec{x}.d\vec{x}$. For a time-like worldline, the interval is negative, and one defines the proper time as $d\tau^2 = -d\vec{x}.d\vec{x}$. Now, consider the vector $d\vec{x}/d\tau$. It has a magnitude, $(d\vec{x}/d\tau).(d\vec{x}/d\tau) = d\vec{x}.d\vec{x}/(d\tau)^2 = -1$. It is also tangent to the worldline since it is a multiple of $d\vec{x}$.

Since in a momentarily comoving reference frame (MCRF), $d\vec{x}$ has components $(dt,0,0,0)$, so that the components of $d\vec{x}/d\tau$ has components $(1,0,0,0)$. This means that in MCRF, $d\vec{x}/d\tau = \vec{e}_0$. But this is the definition of the four-velocity vector. So we write,

$$\vec{U} = \frac{d\vec{x}}{d\tau}. \tag{15}$$

2.3 Scalar product

Keeping in mind the importance of the *interval* (and its Lorentz invariance), we define the (frame invariant) scalar product of two vectors as,

$$\vec{A}.\vec{B} = -A_0 B_0 + A_1 B_1 + A_2 B_2 + A_3 B_3 = A^\alpha B^\beta (\vec{e}_\alpha . \vec{e}_\beta) = A^\alpha B^\beta \eta_{\alpha\beta} , \tag{16}$$

where $\eta_{00} = -1, \eta_{11} = \eta_{22} = \eta_{33} = 1$ and $\eta_{\alpha\beta} = 0, \alpha \neq \beta$. These are called the components of the metric tensor in SR (in Minkowski space). We are now in a position to define a tensor.

2.4 Tensors

We will define a tensor of type $\begin{pmatrix} 0 \\ N \end{pmatrix}$ as a function of N vectors into the real numbers, which is linear in each of its N arguments. The definition of scalar product conforms to this rule (for a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor). The linearity essentially means that

$$(\alpha\vec{A}).\vec{B} = \alpha(\vec{A}.\vec{B}), \qquad (\vec{A}+\vec{B}).\vec{C} = \vec{A}.\vec{C}+\vec{B}.\vec{C}. \tag{17}$$

We introduce a notation for the dot product,

$$\mathbf{g}(\vec{A},\vec{B}) = \vec{A}.\vec{B}, \tag{18}$$

where $\mathbf{g}$ is the metric tensor. The components of this tensor are,

$$\mathbf{g}(\vec{e}_\alpha,\vec{e}_\beta) = \eta_{\alpha\beta}. \tag{19}$$

2.5 One-forms

We would call the $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensors 'one-forms' and denote them with a ˜ overhead, as in $\tilde{p}$. These tensors give a real number when supplied with a vector. That is $\tilde{p}(\vec{A})$ is a number. The components of $\tilde{p}$ would be $p_\alpha \equiv \tilde{p}(\vec{e}_\alpha)$. This would mean that,

$$\tilde{p}(\vec{A}) = \tilde{p}(A^\alpha\vec{e}_\alpha) = A^\alpha\tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha. \tag{20}$$

Notice that all the terms here have plus signs —this is an example of *contraction*.

Let us look at the transformation of one-forms.

$$p_{\overline{\beta}} = \tilde{p}(\vec{e}_{\overline{\beta}}) = \tilde{p}(\Lambda^\alpha_{\overline{\beta}}\vec{e}_\alpha) = \Lambda^\alpha_{\overline{\beta}}p_\alpha. \tag{21}$$

This means that, $p_{\overline{\beta}} = \Lambda^\alpha_{\overline{\beta}}p_\alpha$. Compare this with the transformation rule of basis vectors, $e_{\overline{\beta}} = \Lambda^\alpha_{\overline{\beta}}e_\alpha$.

This also ensures the frame invariance of $A^\alpha p_\alpha$. Remember that (Schutz p.43) for Lorentz transformation, $\Lambda^{\overline{\alpha}}_\beta\Lambda^\mu_{\overline{\alpha}} = \delta^\mu_\beta$. So,

$$A^{\overline{\alpha}}p_{\overline{\alpha}} = (\Lambda^{\overline{\alpha}}_\beta A^\beta)(\Lambda^\mu_{\overline{\alpha}}p_\mu) = \delta^\mu_\beta A^\beta p_\mu = A^\beta p_\beta. \tag{22}$$

We can define the basis one-forms in this way:$\tilde{p} = p_\alpha\tilde{(w)}^\alpha$. But since

$$p_\alpha A^\alpha = \tilde{p}(\vec{A}) = p_\alpha\tilde{w}^\alpha(\vec{A}) = p_\alpha\tilde{w}^\alpha(A^\beta\vec{e}_\beta) = p_\alpha A^\beta\tilde{w}^\alpha(\vec{e}_\beta), \tag{23}$$

so we must have (for the invariance of scalar product), as the β-th component of the α-th basis one-form,

$$\tilde{w}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta. \tag{24}$$

That is, the components of $\tilde{w}^0$ in the frame O are $(1,0,0.0)$ and so on.

Historically, vectors and one-forms have been called contravariant and covariant vectors. The transformation rule of one-forms is the same as that of the basis vectors, and this behaviour of varying *with* the basis vectors, gave one-forms the name *co*variant vectors. They form the dual space of vectors and one-forms. This in similar to 'bra' and 'ket' in the Hilbert space, where for inner product of two function $\psi(x0$ and $\phi(x)$ one needed $\int \psi^*(x)\phi(x)d^3x$.

There is a convenient way of visualising one-forms. One-forms can be thought of as a series of surfaces. The magnitude would depend on the spacing between the surfaces—small magnitude for larger spacings. When a one-form acts on a vector, the resulting number is the number of surfaces pierced by the vector.

That this is consistent with what we have done can be realised easily if we look at the gradient carefully. Suppose we label (parametrize) the world-line by the proper time τ. Suppose $\phi(\vec{x})$ is a scalar field. Then its rate of change along the curve is,

$$\begin{aligned} \frac{d\phi}{d\tau} &= \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau} \\ &= \frac{\partial\phi}{\partial t}U^t + \frac{\partial\phi}{\partial x}U^x + \frac{\partial\phi}{\partial y}U^y + \frac{\partial\phi}{\partial z}U^z . \end{aligned} \tag{25}$$

Notice that this is linear in $\vec{U}$. This means that we can think of a one-form,

$$\tilde{d}\phi \qquad \overrightarrow{O} \qquad (\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}), \tag{26}$$

which when supplied with the vector $\vec{U}$, produces the number $\frac{d\phi}{d\tau}$, the rate of change of ϕ on the curve whose tangent is $\vec{U}$.

The concept of the gradient as a one-form is consistent with the picture we had drawn earlier. In a topographical map, think of one-forms as the contours of constant elevation. Then the change of height as you go in a direction (a vector) is given by the action of the one-form on the vector.

It follows from the definition of the metric tensor that $\mathbf{g}\ (\vec{V},\ )$ is a one-form. Suppose we call it $\tilde{V}(\,)$. Then what are its components?

$$V_\alpha = \tilde{V}(\vec{e}_\alpha) = \vec{V}.\vec{e}_\alpha = \vec{e}_\alpha.\vec{V} = \vec{e}_\alpha.(V^\beta\vec{e}_\beta) = (\vec{e}_\alpha.\vec{e}_\beta)V^\beta = \eta_{\alpha\beta}V^\beta . \tag{27}$$

So that, we have,

$$V_\alpha = \eta_{\alpha\beta}V^\beta . \tag{28}$$

This gives us a mechanism for lowering and raising the indices of tensors.

3 Tensors in polar coordinates

As a prelude to working with curvilinear coordinates, let us work out the properties of tensors in a familiar coordinate, say, the polar coordinates. So, for now we wouldn't talk about SR and will be talking of Euclidean space, and the transformation between Cartesian and the polar coordinates. Here we have,

$$\begin{aligned} r &= \sqrt{x^2+y^2} \qquad x = r\cos\theta\,, && (29)\\ \theta &= \arctan\frac{y}{x} \qquad y = r\sin\theta && (30)\end{aligned}$$

We also have for infinitesimally small increments,

$$\Delta r = \frac{\partial r}{\partial x}\Delta x + \frac{\partial r}{\partial y}\Delta y = \frac{x}{r}\Delta x + \frac{y}{r}\Delta y = \cos\theta\Delta x + \sin\theta\Delta y\,. \tag{31}$$

And similarly,

$$\Delta\theta = \frac{-y}{r^2}\Delta x + \frac{x}{r^2}\Delta y = \frac{-1}{r}\sin\theta\Delta x + \frac{1}{r}\cos\theta\Delta y\,. \tag{32}$$

In general, if we define a coordinate system (with respect to the Cartesian),

$$\begin{aligned} \xi &= \xi(x,y) \qquad \Delta\xi = \frac{\partial\xi}{\partial x}\Delta x + \frac{\partial\xi}{\partial y}\Delta y && (33)\\ \eta &= \eta(x,y) \qquad \Delta\eta = \frac{\partial\eta}{\partial x}\Delta x + \frac{\partial\eta}{\partial y}\Delta y && (34)\end{aligned}$$

The requirement that it is a 'good' coordinate system, that is, $\Delta x = \Delta y = 0$ if $\Delta\xi = \Delta\eta = 0$, means that the Jacobian of the transformation is nonzero. That is,

$$\det\begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{pmatrix} \neq 0 \tag{35}$$

which is the Jacobian of the coordinate transformation.

If we define the above matrix as $\Lambda^{\alpha'}_{\beta}$ then we can define a vector as something that transforms as,

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \tag{36}$$

as in, $V^{\alpha'} = \Lambda^{\alpha'}_{\beta}V^{\beta}$.

The basis vectors in polar coordinates can be easily found out.

$$\begin{aligned} \vec{e}_r &= \Lambda^{\beta}_{r}\vec{e}_{\beta} \\ &= \Lambda^{x}_{r}\vec{e}_x + \Lambda^{y}_{r}\vec{e}_y \\ &= \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y \\ &= \cos\theta\vec{e}_x + \sin\theta\vec{e}_y && (37)\end{aligned}$$

Similarly,

$$\vec{e}_\theta = -r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y\,. \tag{38}$$

Given a scalar field ϕ, we can define a one-form $\tilde{d\phi}$ as a geometrical object with components

$$\tilde{d\phi} \longrightarrow (\partial\phi/\partial\xi, \partial\phi/\partial\eta)\,. \tag{39}$$

This makes sense, since from chain rule of partial differentiation we have,

$$\frac{\partial\phi}{\partial\xi} = \frac{\partial x}{\partial\xi}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial\xi}\frac{\partial\phi}{\partial y} \tag{40}$$

Therefore,

$$\begin{pmatrix} \frac{\partial\phi}{\partial\xi} \\ \frac{\partial\phi}{\partial\eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{pmatrix} \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{pmatrix} \tag{41}$$

Here the conversion matrix is basically $\Lambda^{\alpha}_{\beta'}$ and is the inverse of $\Lambda^{\alpha'}_{\beta}$, as expected.

The basis one-forms are essentially,

$$\begin{aligned} \tilde{d\theta} &= \frac{\partial\theta}{\partial x}\tilde{dx} + \frac{\partial\theta}{\partial y}\tilde{dy} \\ &= -\frac{1}{r}\sin\theta\tilde{dx} + \frac{1}{r}\cos\theta\tilde{dy}\,. \end{aligned} \tag{42}$$

The interesting thing to note here is that the length of the basis vectors in polar coordinates are not constant.

$$|\vec{e}_\theta|^2 = \vec{e}_\theta.\vec{e}_\theta = r^2\sin^2\theta + r^2\cos^2\theta = r^2\,. \tag{43}$$

The metric tensor in polar coordinates would have components,

$$g_{\alpha'\beta'} = \mathbf{g}(\vec{e}_{\alpha'}, \vec{e}_{\beta'}) = \vec{e}_{\alpha'}.\vec{e}_{\beta'}\,, \tag{44}$$

which means that $g_{rr} = 1, g_{\theta\theta} = r^2$ and $g_{r\theta} = 0$. This is best described in writing the expression of a line element,

$$\vec{dl}.\vec{dl} = ds^2 = |dr\vec{e}_r + d\theta\vec{e}_\theta|^2 = dr^2 + r^2 d\theta^2 \tag{45}$$

3.1 Calculus in polar coordinates

The fact that the basis vectors in a general coordinate may change from place to place makes differentiation difficult. If we want to see the change in the component of a vector around a point, we then not only have to differentiate the components, we would also have to consider the fact that the bases would also change. The differentiation of the basis vectors would have to be incorporated too. Let us see what this means with an example in the polar coordinate.

The derivatives of the basis vectors would be as follows:

$$\begin{aligned} \frac{\partial}{\partial r}\vec{e}_r &= \frac{\partial}{\partial r}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = 0 \\ \frac{\partial}{\partial\theta}\vec{e}_r &= \frac{\partial}{\partial\theta}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) \\ &= -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y = \frac{1}{r}\vec{e}_\theta \end{aligned} \tag{46}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial r}\vec{e}_\theta &= \frac{\partial}{\partial r}(-r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y) = \frac{1}{r}\vec{e}_\theta \\
\frac{\partial}{\partial\theta}\vec{e}_\theta &= -r\cos\theta\vec{e}_x - r\sin\theta\vec{e}_y = -r\vec{e}_r\,. \qquad (47)
\end{aligned}$$

Consider the derivative of $\vec{e}_x = \cos\theta\vec{e}_r - \frac{1}{r}\sin\theta\vec{e}_\theta$.

$$\begin{aligned}
\frac{\partial\vec{e}_x}{\partial\theta} &= \frac{\partial}{\partial\theta}(\cos\theta)\vec{e}_r + \cos\theta\frac{\partial}{\partial\theta}(\vec{e}_r) - \frac{\partial}{\partial\theta}(\frac{1}{r}\sin\theta)\vec{e}_\theta - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}(\vec{e}_\theta) \\
&= -\sin\theta\vec{e}_r + \cos\theta\frac{1}{r}\vec{e}_\theta - \frac{1}{r}\cos\theta\vec{e}_\theta - \frac{1}{r}\sin\theta(-r\vec{e}_r) = 0\,, \qquad (48)
\end{aligned}$$

as expected. But notice that the 1st and the 3rd terms here are from differentiating the components of the vector in polar coordinates, whereas the 2nd and the 4th terms come from differentiation of the basis vectors in polar coordinates. In general, we will write,

$$\begin{aligned}
\frac{\partial\vec{V}}{\partial r} &= \frac{\partial}{\partial r}(V^r\vec{e}_r + V^\theta\vec{e}_\theta) \\
&= \frac{\partial V^r}{\partial r}\vec{e}_r + V^r\frac{\partial\vec{e}_r}{\partial r} + \frac{\partial V^\theta}{\partial r}\vec{e}_\theta + V^\theta\frac{\partial\vec{e}_\theta}{\partial r} \\
\frac{\partial}{\partial r}(V^\alpha\vec{e}_\alpha) &= \frac{\partial V^\alpha}{\partial r}\vec{e}_\alpha + V^\alpha\frac{\partial\vec{e}_\alpha}{\partial r} \qquad \alpha = r,\theta \qquad (49)
\end{aligned}$$

And more generally,

$$\frac{\partial\vec{V}}{\partial x^\beta} = \frac{\partial\vec{V}^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha\frac{\partial\vec{e}_\alpha}{\partial x^\beta} = \frac{\partial\vec{V}^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha\Gamma^\mu_{\alpha\beta}\vec{e}_\mu\,, \qquad (50)$$

where $\Gamma^\mu_{\alpha\beta}$ are called the Christoffel symbols They are essentially the μ-th component of $\frac{\partial\vec{e}_\alpha}{\partial x^\beta}$.

In polar coordinates,

$$\begin{aligned}
\frac{\partial\vec{e}_r}{\partial r} &= 0 \qquad \Rightarrow \qquad \Gamma^\mu_{rr} = 0, \text{for all}\,\mu \\
\frac{\partial\vec{e}_r}{\partial\theta} &= \frac{1}{r}\vec{e}_\theta \qquad \Rightarrow \qquad \Gamma^r_{r\theta} = 0, \Gamma^\theta_{r,\theta} = \frac{1}{r} \\
\frac{\partial\vec{e}_\theta}{\partial r} &= \frac{1}{r}\vec{e}_\theta \qquad \Rightarrow \qquad \Gamma^r_{\theta r} = 0, \Gamma^\theta_{\theta r} = \frac{1}{r} \\
\frac{\partial\vec{e}_\theta}{\partial\theta} &= -r\vec{e}_r \qquad \Rightarrow \qquad \Gamma^r_{\theta\theta} = -r, \Gamma^\theta_{\theta\theta} = 0\,. \qquad (51)
\end{aligned}$$

In the equation (50) if we change the dummy indices $\mu \to \alpha$ and $\alpha \to \mu$, then we can write,

$$\frac{\partial\vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu\Gamma^\alpha_{\mu\beta}\right)\vec{e}_\alpha\,. \qquad (52)$$

We introduce a few new notations and write this as,

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + V^\mu\Gamma^\alpha_{\mu\beta}\,. \qquad (53)$$

We will call $\frac{\partial\vec{V}}{\partial x^\beta} = V^\alpha_{;\beta}\vec{e}_\alpha \equiv \nabla\vec{V}$ the covariant derivative of $\vec{V}$.

Similarly, we can define the divergence as $V^{\alpha}_{;\alpha}$. In polar coordinates,

$$\begin{aligned}\Gamma^{\alpha}_{r\alpha} &= \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} = \frac{1}{r} \\ \Gamma^{\alpha}_{\theta\alpha} &= \Gamma^{r}_{\theta r} + \Gamma^{\theta}_{\theta\theta} = 0 \,.\end{aligned} \tag{54}$$

Therefore, the divergence is,

$$V^{\alpha}_{;\alpha} = \frac{\partial V^r}{\partial r} + \frac{\partial V^{\theta}}{\partial \theta} + \frac{1}{r}V^r = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial\theta}V^{\theta} \,, \tag{55}$$

which is the familiar expression.

The covariant derivative of one-forms are abit different. We use the fact that a one-form $\tilde{p}$ gives a number ϕ when acting on a vector $\vec{V}$. The derivative of ϕ (which is its covariant derivative since ϕ does not depend on the basis vectors) is,

$$\begin{aligned}\nabla_{\beta}\phi &= \nabla_{\beta}(p_{\alpha}V^{\alpha}) \\ &= \frac{\partial p_{\alpha}}{\partial x^{\beta}}V^{\alpha} + p_{\alpha}\frac{\partial V^{\alpha}}{\partial x^{\beta}} \\ &= \frac{\partial p_{\alpha}}{\partial x^{\beta}}V^{\alpha} + p_{\alpha}V^{\alpha}_{;\beta} - p_{\alpha}V^{\mu}\Gamma^{\alpha}_{\mu\beta} \\ &= \left(\frac{\partial p_{\alpha}}{\partial x^{\beta}} - p_{\mu}\Gamma^{\mu}_{\alpha\beta}\right)V^{\alpha} + p_{\alpha}V^{\alpha}_{;\beta} \,,\end{aligned} \tag{56}$$

Therefore, if we set $(\nabla_{\beta}\tilde{p})_{\alpha} = (\nabla\tilde{p})_{\alpha\beta} \equiv p_{\alpha;\beta} = p_{\alpha,\beta} - p_{\mu}\Gamma^{\mu}_{\alpha\beta}$, then we could write, $\nabla_{\beta}(p_{\alpha}V^{\alpha}) = p_{\alpha;\beta}V^{\alpha} + p_{\alpha}V^{\alpha}_{;\beta}$. Therefore,

$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_{\mu}\Gamma^{\mu}_{\alpha\beta} \,. \tag{57}$$

---

4 Curved spacetime

We are now in a position to discuss curved space. One talks of a *manifold* which is basically a continuous space which looks locally like Euclidean space. For example the surface of a sphere is a manifold. And so is any m-dimensional 'hyperplane' in an n- dimensional Euclidean space ($m \leq n$). Essentially a manifold is any set that can be continuously parametrized, and the number of independent parameters is the dimension of the manifold and the parameters are the coordinates.

Here we will only talk about differentiable manifolds, which are continuous and differentiable. And we would also like to have a definition of a metric tensor in this manifold. Such manifolds are called Riemannian manifolds. Notice that we have now added some 'structure' to the manifold by adding the definition of the metric tensor – a definition for distances and time differences.

Now it turns out that curved spaces are locally flat. In our context, this means that at any point one can find a flat Minkowskian frame, that is a locally inertial frame, in which laws of SR hold. Notice that we are saying that this is possible only locally, in a small region of spacetime.

Mathematically, this basically means that one can find a frame near any point $\mathcal{P}$ in which the metric tensor has the components $g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta}$ for all α, β, and also that

$g_{\alpha\beta,\gamma} = 0$ for all α, β, γ. That it is always possible can be proved by looking at the number of components of $g_{\alpha\beta}$ and the number of constraints that the above equations imply.

Suppose we want to find the components of $g_{\alpha\beta}$ in a different system. The transformation matrix in the equation,

$$g_{\bar{\alpha}\bar{\beta}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}} g_{\alpha\beta} \tag{58}$$

has 16 coefficients. This means 16 equations. On the other hand, the number of independent components in $g_{\alpha\beta}$ is 10, as it is a symmetric tensor. So, there are enough free parameters to transform the metric tensor so that it can be put equal to the Minkowski form at $\mathcal{P}$. Now, the first derivatives, $g_{\alpha\beta,\gamma}$ has 4×10 components. This means that the combined condition that the metric tensor is equal to the Minkowski form and that its first derivatives vanish locally, has fifty conditions. How many equations do we have? The transformation law for $g_{\alpha\beta,\gamma}$ involves the above mentioned 16 transformation coefficients $\frac{\partial x^{\alpha}}{\partial x^{\bar{\beta}}}$ and their forty derivatives $\frac{\partial^2 x^{\alpha}}{\partial x^{\bar{\beta}} x^{\bar{\gamma}}}$. So, we have fifty six numbers and fifty conditions to satisfy. This leaves room for three free spatial rotations and three Lorentz velocity transformations.

So, we see that we can satisfy the above conditions, that is the metric is locally Minkowskian and its first derivatives vanish there. Notice that in general we cannot put the second derivatives to zero, as this would involve more equations than there are numbers available.

What does this mean physically? We have essentially said that at any point one can find a local inertial frame, in which laws of SR hold, and the vanishing of the first derivatives of the metric tensor means that locally it is a flat space. This is essentially the *Principle of Equivalence*. I will quote from Weinberg the formulation of the principle as *"that at every space-time point in an arbitrary gravitational field it is possible to choose a 'locally inertial coordinate system' such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation"*.

That we can do that actually due to the fact that the inertial and graviational masses are equal. Einstein thought that this leads to the conclusion that in a free-falling elevator, for example, no homogeneous gravitational field can be detected. In other words, for a particle in a graviational field, such as near the Earth, one can always find a frame, the free-falling elevator, in which the particle would move according to the laws of nature as if there were no gravitational field. Notice that it is only possible locally. Since the graviational field near Earth is inhomogeneous, the particles would move closer in the elevator as it falls down, revealing the gravitational field.

One talks of weak and strong equivalence principles in this context. The weak principle involves only gravity. The strong equivalence principle means 'all laws of nature' when we talk of particles following laws of nature.

Experiments by Eötvös and Dicke and others provide direct evidence for the weak principle of equivalence, and hints for the strong principle, since they involve objects of different compositions, with different ratios of neutrons and protons, say. They have shown that the principle of equivalence is true within a very high accuracy (almost one part in 10^{11}).

Now let us go back to covariant differentiation. For this, we would have to take the difference of vectors at two different points. In a curved space, however, this difference should be done carefully since in between the points the space is curved and the concept of the vectors at the two points, pointing in the 'same' direction is vague. But since locally we can approximate the space as being flat, we can take the difference in the limit as they get infinitesimally close together. This leads to some important results.

4.1 Christoffel symbols and the metric

The relation between Christoffel symbols and the metric are very important for calculations and we will derive them now. Now, the local inertial frame is a frame in which SR laws hold, and in SR the derivatives of basis vectors are zero, that is Christoffel symbols vanish. This means that for the metric tensor, $g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma}$ which is equal to zero at $\mathcal{P}$. But since this is a valid tensor equation, $g_{\alpha\beta;\gamma} = 0$ at $\mathcal{P}$ in any basis.

Before we can express the Christoffel symbols in terms of the metric, we must prove that $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$. Consider an arbitrary scalar field ϕ. Its first derivative $\nabla\phi$ is a one-form with components $\phi_{,\beta}$. Its second covariant derivative $\nabla\nabla\phi$ has components $\phi_{,\beta;\alpha}$. But locally, this is equal to $\phi_{,\beta,\alpha}$, which is equal to $\phi_{,\alpha,\beta}$ as the partial derivatives commute. So the tensor $\phi_{,\beta,\alpha}$ is symmetric in the local inertial frame, but if it is symmetric in one frame it must be symmetric in all frames. So,

$$\phi_{,\beta;\alpha} = \phi_{,\alpha;\beta}\,. \tag{59}$$

But

$$\phi_{,\beta;\alpha} = \phi_{,\beta,\alpha} - \phi_{,\mu}\Gamma^{\mu}_{\beta\alpha} = \phi_{,\alpha;\beta} = \phi_{,\alpha,\beta} - \phi_{,\mu}\Gamma^{\mu}_{\alpha\beta} \tag{60}$$

which leads to the result that $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$.

Now, from eqn (57) we can write,

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} - \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu}\,. \tag{61}$$

This can be manipulated to write a series of equations,

$$\begin{aligned} g_{\alpha\beta,\mu} &= \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu} \\ g_{\alpha\mu,\beta} &= \Gamma^{\nu}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta}g_{\alpha\nu} \\ g_{\beta\mu,\alpha} &= \Gamma^{\nu}_{\beta\alpha}g_{\nu\mu} + \Gamma^{\nu}_{\mu\alpha}g_{\beta\nu}\,, \end{aligned} \tag{62}$$

We sum these up and using the symmetry $g_{\alpha\beta} = g_{\beta\alpha}$, we get

$$\begin{aligned} g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= g_{\nu\beta}(\Gamma^{\nu}_{\alpha\mu} - \Gamma^{\nu}_{\mu\alpha}) + g_{\nu\mu}(\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\beta\alpha}) + g_{\alpha\nu}(\Gamma^{\nu}_{\beta\mu} + \Gamma^{\nu}_{\mu\beta}) \\ &= 2g_{\alpha\nu}\Gamma^{\nu}_{\beta\mu}\,. \end{aligned} \tag{63}$$

This leads to (after multiplying by $g^{\alpha\gamma}$ and using $g^{\alpha\gamma}g_{\alpha\nu} = \delta^{\gamma}_{\nu}$),

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\,. \tag{64}$$

4.2 Parallel transport

We would like to quantify the curvature of space, and it is easy to do it if we develop the concept of parallel transport. Consider a triangle in flat space (with sides which are not necessarily straight lines). Suppose at point A we start with a vector and try to transport this vector around the triangle back to A. We will end up with the same vector. But consider now a triangle on the surface of a sphere. Suppose B is the at the pole and A and C are at the equator. We will find that as we try to transport a vector around the loop, trying to keep the vector as close to parallel as possible to its counterpart at a neighbouring place, then we would not get the same vecor at the end of the loop. The difference is actually a measure of the curvature.

One must make clear at this point that we are talking of 'intrinsic' curvature. The surface of a cylinder is actually flat — it looks curved because of the fact that the two dimensional flat surface is embedded in three dimensional space. We would call this sort of curvature the extrinsic curvature.

Suppose we parametrize a curve by the arc length s , the tangent to the curve is basically a vector $\vec{U} = \frac{d\vec{x}}{ds}$ where $\vec{U}$ is not necessarily normalized. In a locally inertial frame at a point $\mathcal{P}$, the parallel transport of a vector $\vec{V}$ would then mean that $\frac{dV^\alpha}{ds} = 0 = U^\beta V^\alpha_{,\beta}$. But in the locally inertial frame, $U^\beta V^\alpha_{,\beta} = U^\beta V^\alpha_{;\beta}$. But since this is a valid tensor equation, this must be true in all basis, and do it can be taken as the frame invariant definition of parallel transport of vector $\vec{V}$ along the curve with tangent $\vec{U}$. We will write,

$$\frac{dV^\alpha}{ds} = U^\beta V^\alpha_{;\beta} \equiv \nabla_{\vec{U}} \vec{V} = 0\,. \tag{65}$$

Or, in other words,

$$U^\beta V^\alpha_{,\beta} = -\Gamma^\alpha_{\mu\beta} V^\mu U^\beta \tag{66}$$

that is,

$$V^\alpha_{,\beta} = -\Gamma^\alpha_{\mu\beta} V^\mu\,. \tag{67}$$

Now, in Cartesian coordinates, a straight line can be defined as a curve which parallel transports its own tangent. In a general curved space, we can define 'straight lines', which would call *geodesics*, as curves which parallel transports their own tangents. That is,

$$U^\beta U^\alpha_{;\beta} = 0 = U^\beta U^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} U^\mu U^\beta = 0\,. \tag{68}$$

Or, if one writes, $U^\alpha = \frac{dx^\alpha}{ds}$ and $U^\beta \frac{\partial}{\partial x^\beta} = \frac{d}{ds}$, then

$$\frac{d}{ds}\frac{dx^\alpha}{ds} + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{ds}\frac{dx^\beta}{ds}\,. \tag{69}$$

4.3 Curvature tensor

We would like to get a measure of the curvature by taking a vector around a closed loop, that is, parallel transport it around a loop. Consider a small loop in the manifold, whose four sides are the lines $x^1 = a, x^1 = a + \delta a, x^2 = b, x^2 = b + \delta b$. A vector $\vec{V}$ is first transported from A to B. From the equation for parallel transport, we have

$$\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1} V^\mu\,. \tag{70}$$

Therefore, at B, the vector will be,

$$
\begin{aligned}
V^{\alpha}(B) &= V^{\alpha}(A) + \int_A^B \frac{\partial V^{\alpha}}{\partial x^1} dx^1 \\
&= V^{\alpha}(A) - \int_{x^2=b} \Gamma^{\alpha}_{\mu 1} V^{\mu} dx^1
\end{aligned}
\tag{71}
$$

where the label $x^2 = b$ means the path AB. Similarly transporting it to C and then to D gives,

$$
\begin{aligned}
V^{\alpha}(C) &= V^{\alpha}(B) - \int_{x^1=a+\delta a} \Gamma^{\alpha}_{\mu 2} V^{\mu} dx^2 \\
V^{\alpha}(D) &= V^{\alpha}(C) + \int_{x^2=b+\delta b} \Gamma^{\alpha}_{\mu 1} V^{\mu} dx^1
\end{aligned}
\tag{72}
$$

The last integral is negative because the path is traversed in the opposite x^1 direction. Finally when we arrive at A, we have

$$
V^{\alpha}(A_{final}) = V^{\alpha}(D) + \int_{x^1=a} \Gamma^{\alpha}_{\mu 2} V^{\mu} dx^2 \,. \tag{73}
$$

So, the net change is approximately,

$$
\begin{aligned}
\delta V^{\alpha} &= V^{\alpha}(A_{final}) - V^{\alpha}(A_{initial}) \\
&\approx \int_{x^1=a} \Gamma^{\alpha}_{\mu 2} V^{\mu}) dx^2 - \int_{x^1=a+\delta a} \Gamma^{\alpha}_{\mu 2} V^{\mu}) dx^2 + \int_{x^2=b+\delta b} \Gamma^{\alpha}_{\mu 1} V^{\mu}) dx^1 - \int_{x^2=b} \Gamma^{\alpha}_{\mu 1} V^{\mu}) dx^1 \\
&\approx -\int_b^{b+\delta b} \delta a \frac{\partial}{\partial x^1} (\Gamma^{\alpha}_{\mu 2} V^{\mu}) dx^2 + \int_a^{a+\delta a} \delta b \frac{\partial}{\partial x^2} (\Gamma^{\alpha}_{\mu 1} V^{\mu}) dx^1 \\
&\approx \delta a \delta b \left(-\frac{\partial}{\partial x^1} (\Gamma^{\alpha}_{\mu 2} V^{\mu}) + \frac{\partial}{\partial x^2} (\Gamma^{\alpha}_{\mu 1} V^{\mu}) \right) \\
&\approx \delta a \delta b (\Gamma^{\alpha}_{\mu 1,2} - \Gamma^{\alpha}_{\mu 2,1} + \Gamma^{\alpha}_{\nu 2} \Gamma^{\nu}_{\mu 1} - \Gamma^{\alpha}_{\nu 1} \Gamma^{\nu}_{\mu 2}) V^{\mu} \,.
\end{aligned}
\tag{74}
$$

One defines the Riemann Curvature Tensor $R^{\alpha}_{\mu\lambda\sigma}$ as,

$$
R^{\alpha}_{\mu\lambda\sigma} = \Gamma^{\alpha}_{\mu\sigma,\lambda} - \Gamma^{\alpha}_{\mu\lambda,\sigma} + \Gamma^{\alpha}_{\nu\lambda} \Gamma^{\nu}_{\mu\sigma} - \Gamma^{\alpha}_{\nu\sigma} \Gamma^{\nu}_{\mu\lambda} \,. \tag{75}
$$

This is a $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ tensor that gives δV^{α} when supplied with $\vec{V}, \delta a \vec{e}_{\sigma}, \delta b \vec{e}_{\lambda}$. Using the relation between the Christoffel symbols and the metric in the last section, we can write the curvature tensor in terms of the components of metric tensor. Locally, we can say that the Christoffel symbols vanish, although their first derivatives do not. Since the derivatives are,

$$
\Gamma^{\alpha}_{\mu\nu,\sigma} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu\sigma} + g_{beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma}) \,. \tag{76}
$$

we have,

$$
R^{\alpha}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} (g_{\sigma\beta,\nu\mu} + g_{\sigma\nu,\beta\mu} - g_{\beta\nu,\sigma\mu} - g_{\sigma\beta,\mu\nu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu}) \tag{77}
$$

so that finally we have (from symmetry of the metric tensor components),

$$
R^{\alpha}_{\beta,\mu\nu} = \frac{1}{2} g^{\alpha\sigma} (g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}) \,. \tag{78}
$$

If we define $R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\mu\nu}$, then one can show,

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0\,. \tag{79}$$

Also,

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}\,. \tag{80}$$

This means that $R_{\alpha\beta\mu\nu}$ is antisymmetric on the first pair and on the second pair but symmetric on exchange of the two parts.

We can contract the first and third index and get the Ricci tensor, $R_{\beta\nu} = g^{\alpha mu} R_{\alpha\beta\mu\nu} = R^{\mu}_{\beta\mu\nu}$, which is a symmetric tensor. The Ricci scalar is derived by contracting further, $R = g^{\beta\nu} R_{\beta\nu}$.

We now derive the expression for the Einstein tensor. In a locally inertial (Minkowski) frame,

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda})\,. \tag{81}$$

Using the symmetry of **g**, one can write,

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0\,. \tag{82}$$

In general, one writes,

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0\,. \tag{83}$$

This is called the Bianchi indentity. Let us apply the Ricci contraction to this identity (multiply by $g^{\alpha\mu}$). The first term is $g^{\alpha\mu} R_{\alpha\beta\mu\nu;\lambda} = R_{\beta\nu;\lambda}$. The second term is $g^{\alpha\mu} R_{\alpha\beta\lambda\mu;\nu} = -g^{\alpha\mu} R_{\alpha\beta\mu\lambda;\nu} = -R_{\beta\lambda;\nu}$. The last term is $g^{\alpha\mu} R_{\alpha\beta\nu\lambda;\mu} = R^{\mu}_{\beta\nu\lambda;\mu}$. So we get,

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}_{\beta\nu\lambda;\mu} = 0\,. \tag{84}$$

We contract again by multiplying by $g^{\beta\nu}$. We get,

$$R_{;\lambda} - R^{\mu}_{\lambda;\mu} + (-R^{\mu}_{\lambda;\mu} = 0\,. \tag{85}$$

Here we have used the fact that

$$g^{\beta\nu} R^{\mu}_{\beta\nu\lambda;\mu} = g^{\beta\nu} g^{\mu\alpha} R_{\alpha\beta\nu\lambda;\mu} = -g^{\beta\nu} g^{\mu\alpha} R_{\beta\alpha\nu\lambda;\mu} = g^{\mu\alpha} R_{\alpha\lambda;\mu} = -R^{\mu}_{\lambda;\mu}\,. \tag{86}$$

Now we can write the equation as,

$$(2R^{\mu}_{\lambda} - \delta^{\mu}_{\lambda} R)_{;\mu} = 0\,. \tag{87}$$

Multiplying by $g^{\lambda\sigma}$, this can be written as,

$$(R^{\mu\sigma} - \frac{1}{2} g^{\mu\sigma} R)_{;\mu} = (R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R)_{;\alpha} = 0\,. \tag{88}$$

And by defining the Einstein tensor $G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = G^{\beta\alpha}$, one can write,

$$G^{\alpha\beta}_{;\alpha} = 0\,. \tag{89}$$

4.4 Toward Einstein's equation – Stress energy tensor

In the Newtonian theory the source of gravity is the mass density ρ. It is tempting to say that in a relativistic theory of gravity, the source term should be the mass and energy density. But this quantity mn ($c = 1$, and n is the number density of particles) varies as $mn \longrightarrow \frac{m}{\sqrt{1-v^2}} \frac{n}{\sqrt{1-v^2}}$. The last term is because of the length contraction in the direction of the velocity and the resulting increase in number density. Now, this transformation involves two factors of $\Lambda^{\bar{o}}_{o} = \frac{1}{\sqrt{1-v^2}}$, like the component of a tensor. Probably it *is* a component of a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor.

Before we can discuss a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor that describes the matter-energy content in a region, let us first look at a related $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ tensor. Consider the flux of particles across surfaces defined by $x^\alpha =$constant, that is number crossing per unit area per unit time. If in the lab frame we have a bunch of particles (with rest density n) all with x-velocity v, then the flux across a surface of constant x is $\frac{nv}{\sqrt{1-v^2}}$. We can easily convince ourselves that, in general, the flux of particles with velocity $\vec{v}$ across a surface $x^\alpha =$constant is nU^α. If $\alpha = 0$ (the flux across a constant time surface), we have the density of particles. Let us try to build a higher order tensor which is similar to this. Instead of flux of particles, we consider the flux of momenta.

We define this Stress-energy tensor as having components $T^{\alpha\beta}$ which is the flux of α- momentum across a surface of constant x^β. Mathematically, we can write it as,

$$T^{\alpha\beta} = \rho U^\alpha U^\beta. \tag{90}$$

Let us see what this means for different systems.

(1) Motionless dust: A *dust* is a collection of particles with straight, mutually parallel worldlines. If the worldlines are parallel to the time axis of a certain inertial observer, then for him the dust is motionless. In this case, there is no momentum, and therefore, only one element of $\mathbf{T}$ is nonzero: $T^{00} = \rho =$rest mass density.

(2) Dust with fixed velocity: In this case the tensor would have components,

$$T^{\alpha\beta} = \rho \frac{1}{1-v^2} \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & (v^1)^2 & v^1 v^2 & v^1 v^3 \\ v^2 & v^2 v^1 & (v^2)^2 & v^2 v^3 \\ v^3 & v^3 v^1 & v^3 v^2 & (v^3)^2 \end{bmatrix}. \tag{91}$$

(3) Monoenergetic gas: Here the particles move with the same speed v, but their directions of motion are random and the particles are not considered to interact in any way. The stress energy tensor for this gas can be found by averaging the previous one over all directions. Which means that terms like v^1 will average out to zero, while the average of $(v^1)^2$ etc. will be $v^2/3$. That is, (in MCRF)

$$T^{\alpha\beta} = \rho \frac{1}{1-v^2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & v^2/3 & 0 & 0 \\ 0 & 0 & v^2/3 & 0 \\ 0 & 0 & 0 & v^2/3 \end{bmatrix}. \tag{92}$$

Here we see the appearance of an isotropic pressure $p = \frac{1}{3}\rho \frac{1}{1-v^2} v^2$.

(4) Stationary general gas: Again the particles are noninteracting, but now they have a distribution of speed v. Averaging the matter tensor the previous case over the distribution one gets (in MCRF),

$$T^{\alpha\beta} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \tag{93}$$

Or, $T^{\alpha\beta} = (\rho + p)U^{\alpha}U^{\beta} + p\eta^{\alpha\beta}$.

So, **T** represents the energy and momentum content of a gas. This means that there must be some law of conservation for **T**. Consider a fluid element (say, in two space directions, at a constant time). The rate of flow of energy across the face with $x = 0$ is $l^2 T^{0x}$, and across the face $x = l$ is $-l^2 T^{0x}$, since it would be in the opposite direction (we want the flow into the fluid element). Similarly, the energy flowing in the y-direction is $l^2 T^{0y}(y=0) - l^2 T^{0y}(y=l)$. The sum of these rates must be the rate of increase of energy inside, $\partial(T^{00} l^3)/\partial t$. Therefore we have (dividing by l^3 and taking the limit $l \to 0$,

$$\frac{\partial}{\partial t} T^{00} = -\frac{\partial}{\partial x} T^{0x} - \frac{\partial}{\partial y} T^{0y} - \frac{\partial}{\partial z} T^{0z}. \tag{94}$$

Or, in other words,

$$T^{00}_{,0} + T^{0x}_{,x} + T^{0y}_{,y} + T^{0z}_{,z} = T^{0\alpha}_{,\alpha} = 0. \tag{95}$$

This is essentially the law of conservation of energy. Similarly from the conservation of momentum, one gets in general,

$$T^{\alpha\beta}_{,\beta} = 0. \tag{96}$$

This is what is expected from SR. Now since in the locally inertial frame we can convert the comma to semicolon, so from equivalence principle, it would be true in all frames. So, we can write,

$$T^{\alpha\beta}_{;\beta} = 0. \tag{97}$$

4.5 Einstein's equation

Comparing the two equations $G^{\alpha\beta}_{;\beta} = 0$ and $T^{\alpha\beta}_{;\beta} = 0$ it is tempting to say that they are proportional. We can write **G**=k **T** where k is some constant. This is not the most general equation that one can write though. In general, one would like an expression involving the metric to be equal to **T**. This expression should be a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor. Ricci tensor is one such tensor. The most general equation that one can write is,

$$F^{\alpha\beta} = R^{\alpha\beta} + \mu g^{\alpha\beta} R + \Lambda g^{\alpha\beta}, \tag{98}$$

where μ, Λ are constants. To determine μ we use the law of conservation of **T**. This means that $F^{\alpha\beta}_{;\beta} = 0$. Since $g^{\alpha\beta}_{;\mu} = 0$, we have,

$$(R^{\alpha\beta} + \mu g^{\alpha\beta} R)_{;\beta} = 0. \tag{99}$$

From the definition of **G** we know that this is possible only when $\mu = -\frac{1}{2}$. Therefore,

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta}\,, \tag{100}$$

where k, Λ are constants. Without the indices, this is **G**+ Λ **g** = **T**.

We will come to the Λ term when we discuss cosmology, since it was not originally present in Einstin's equation, and was introduced by him later, and for the time being we will just use,

$$G^{\alpha\beta} = 8\pi(G/c^2)T^{\alpha\beta}\,, \tag{101}$$

where we have put $k = 8\pi(G/c^2)$. We will soon see that this is needed to make the relativistic theory of gravitation consistent with the Newtonian theory in the limit.

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5 Application of Einstein's equation

5.1 Weak field limit

In the limit of weak fields, we assume that **g** is almost Minkowskian; that is,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}\,, \tag{102}$$

where $h_{\alpha\beta}$ are corrections to the Minkowski tensor $\eta_{\alpha\beta}$. We will try to compute the gravitational field equations to first order in $h_{\alpha\beta}$. This is the weak field limit.

To first order in $h_{\alpha\beta}$, the Ricci tensor is,

$$R_{\lambda\sigma} = \frac{1}{2}\eta^{\alpha\mu}(h_{\lambda\mu,\alpha\sigma} + h_{\alpha\sigma,\lambda\mu} - h_{\lambda\sigma,\alpha\mu} - h_{\alpha\mu,\lambda\sigma})\,, \tag{103}$$

since the other terms contain products of Christoffel symbols and are of order h^2. Rearranging, we can write (and writing $\Box\phi = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi = \eta^{\alpha\mu}\phi_{,\alpha\mu}$),

$$R_{\lambda\sigma} = \frac{1}{2}\left[\left(h^{\alpha}_{\lambda,\alpha} - (1/2)h_{,\lambda}\right)_{,\sigma} + \frac{\partial}{\partial x^{\lambda}}\left(h^{\alpha}_{\sigma,\alpha} - 1/2)h_{,\sigma}\right) - \Box h_{\lambda\sigma}\right]\,, \tag{104}$$

where we have used the Minkowski tensor to raise and lower the indices, $h^{\alpha}_{\lambda} = \eta^{\alpha\mu}h_{\mu\lambda}$, $h = \eta^{\alpha\mu}h_{\alpha\mu}$.

Now, we can change the h^{α}_{β} by a coordinate transformation $\overline{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}$. This gives four free functions $\xi^{\alpha}(x)$. We can choose them so as to satisfy the four conditions,

$$h^{\alpha}_{\lambda,\alpha} - (1/2)h_{,\lambda} = 0\,. \tag{105}$$

This is called the Lorentz Gauge. By doing this, we can write,

$$R_{\lambda\sigma} = -\frac{1}{2}\Box h_{\lambda\sigma}\,. \tag{106}$$

Now, we can rewrite the Einstein equation $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi GT_{\alpha\beta}$, by taking the trace of the equation,

$$R = -8\pi GT\,, \tag{107}$$

where $T = T^{\alpha}_{\alpha}$ is the trace of the stress-energy tensor, and we have used $g_{\alpha\beta}g^{\alpha\beta} = \delta^{\alpha}_{\alpha} = 4$. That is,

$$R_{\alpha\beta} = 8\pi G(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)\,. \tag{108}$$

Therefore, we can write, in the weak field limit,

$$\Box h_{\alpha\beta} = -16\pi G(T_{\alpha\beta} - \eta_{\alpha\beta}T/2)\,. \tag{109}$$

Notice that this is basically a wave equation and gives rise to what are called the gravitational waves.

Now, in the nonrelativistic limit $T_{00}(=\rho)$ is the most dominant term and $T = g^{\alpha\beta}T_{\alpha\beta} = g^{00}T_{00} = -\rho$. And we will also ignore gravitational radiation and say that the time derivatives of $h_{\alpha\beta}$ are small compared to its space derivatives. So that we can write (noting that $\Box \to \nabla^2$ if one drops the time derivative, as $\Box \equiv -\frac{\partial}{\partial t^2} + \nabla^2$),

$$\nabla^2 h_{00} = -8\pi G\rho, \qquad \nabla^2 h_{ij} = -8\pi G\rho\delta_{ij}, \qquad \nabla^2 h_{0i} = 0\,. \tag{110}$$

Comparing these with Poisson's equation $\nabla^2\phi = 4\pi G\rho$, ϕ is the Newtonian potential, we can write,

$$h_{00} = -2\phi, \qquad h_{ij} = -2\delta_{ij}\phi, \qquad h_{0i} = 0\,. \tag{111}$$

So that $g_{00} = -1-2\phi$, $g_{ij} = 1-2\phi$. Therefore, in the weak field limit, the interval can be written as,

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)\,. \tag{112}$$

Let us look at the equation of motion of particles in this weak field limit. Since ϕ is small, we will work out things to first order in ϕ. For a freely falling particle, the four-momentum vector is $\vec{p} = m\vec{U}$, where $\vec{U} = d\vec{x}/d\tau$. Since the path of the freely falling particle is to be a geodesic, which is given by,

$$\nabla_{\vec{U}}\vec{U} = 0\,. \tag{113}$$

Since any constant times the proper time can also be a parameter for labeling the world-line, we can also write the equation in terms of the momentum vector as,

$$\nabla_{\vec{p}}\vec{p} = 0\,. \tag{114}$$

Now, in the non-relativistic limit, the time component of this equation is,

$$p^\alpha p^0_{,\alpha} + \Gamma^0_{\alpha\beta}p^\alpha p^\beta = 0\,. \tag{115}$$

Since in the non-relativistic limit $p^0 \gg p^1$, and since $p^\alpha\partial_\alpha = mU^\alpha\partial_\alpha = md/d\tau$, we have,

$$m\frac{d}{d\tau}p^0 + \Gamma^0_{00}(p^0)^2 = 0\,. \tag{116}$$

But,

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}g^{0\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha}) = \frac{1}{2}g^{00}g_{00,0} = \frac{1}{2}\frac{1}{-(1+2\phi)}(-2\phi_{,0}) \\
&\approx \phi_{,0} + 0(\phi^2)\,.
\end{aligned} \tag{117}$$

We had used the fact that since $[g_{\alpha\beta}]$ is diagonal, $[g^{\alpha\beta}]$ is also diagonal and its elements are the reciprocals of $[g_{\alpha\beta}]$. So, $g^{0\alpha}$ is nonzero only when $\alpha = 0$. Therefore, to first order in ϕ,

$$\frac{d}{d\tau}p^0 = -m\frac{\partial\phi}{\partial t}\,. \tag{118}$$

This basically means that energy is conserved unless the gravitational field depends on time, which is consistent with the Newtonian theory.

The spatial components of the geodesic equation are,

$$p^\alpha p^i_{,\alpha} + \Gamma^i_{\alpha\beta} p^\alpha p^\beta = 0\,. \tag{119}$$

Or, to the lowest order in velocity,

$$m\frac{dp^i}{d\tau} + \Gamma^i_{00}(p^0)^2 = 0\,, \tag{120}$$

neglecting p^i in the summation of Γ. We put $(p^0)^2 \approx m^2$, and write,

$$\frac{dp^i}{d\tau} = -m\Gamma^i_{00}\,. \tag{121}$$

Now,

$$\Gamma^i_{00} = \frac{1}{2} g^{i\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha})\,. \tag{122}$$

Again, since $[g^{\alpha\beta}]$ is diagonal, we have $g^{i\alpha} = (1-2\phi)^{-1}\delta^{i\alpha}$. This gives us,

$$\Gamma^i_{00} = \frac{1}{2}(1-2\phi)^{-1}\delta^{ij}(2g_{j0,0} - g_{00,j})\,. \tag{123}$$

(We have changed α to j as $\delta^{i0} = 0$.) But since $g_{j0} = 0$, we have,

$$\begin{aligned} \Gamma^i_{00} &= -\frac{1}{2} g_{00,j}\delta^{ij} + 0(\phi^2) \\ &= -\frac{1}{2}(-2\phi)_{,j}\delta^{ij}\,. \end{aligned} \tag{124}$$

And the equation of motion is then,

$$\frac{dp^i}{d\tau} = -m\phi_{,j}\delta^{ij}\,. \tag{125}$$

This is the usual Newtonian theory, since the force of a gravitational field is $-m\nabla\phi$. So we see that we have recovered the Newtonian theory in the weak field limit.

5.2 Spherically symmetric metrics: Schwarzschild metric

We will derive the metric for a centrally symmetric metric. If we use the 'spherical' space coordinates, r,θ,ϕ, then the most general centrally symmetric metric that one can write is,

$$ds^2 = h(r,t)dr^2 + k(r,t)(\sin^2\theta d\phi^2 + d\theta^2) + l(r,t)dt^2 + a(r,t)dr\,dt\,, \tag{126}$$

where a,h,k,l are some functions of r and t. By the way, in Euclidean geometry we would have called a metric centrally symmetric if the metric is the same for all points located at the same distance from the center, that is, with the same radius vector. But in curved space-time, there is nothing like a 'radius vector', that is, there is no quantity which gives the distance from the centre *and* is equal to the circumference divided by 2π. So, our choice of the 'radius vector' is arbitrary. So that we can transform the coordinates r,t without destroying the central symmetry of ds^2 as,

$$r = f_1(r',t'), \qquad t = f_2(r',t')\,, \tag{127}$$

where f_1, f_2 are some functions.

Now, we use this possibility to choose the coordinate r and time t so that the coefficients $a(r,t)$ vanishes and $k(r,t) = r^2$. The last condition means that r is defined in such a way that the circumference of a circle with centre at the origin of coordinates is equal to $2\pi r$, since the element of arc of a circle in the plane $\theta = \pi/2$ is equal to $dl = rd\phi$. We will write the functions h,l in exponential form, as $-\exp(2\Phi)$ and $\exp(2\Lambda)$ respectively. So that we have,

$$ds^2 = -\exp(2\Phi)dt^2 + \exp(2\Lambda)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{128}$$

We will simplify the calculations somewhat easier by assuming that Φ,Λ are independent of time, that is the metric is a static one. However, it can be proved that a centrally symmetric metric is always static.

The metric components are,

$$g_{00} = -\exp(2\Phi) \quad ,g_{rr} = \exp(2\Lambda), \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta\,. \tag{129}$$

Clearly,

$$g^{00} = -\exp(-2\Phi) \quad ,g^{rr} = \exp(-2\Lambda), \quad g^{\theta\theta} = r^{-2}, \quad g^{\phi\phi} = r^{-2}\sin^{-2}\theta\,. \tag{130}$$

The non-zero Christoffel symbols are then (denoting differentiation with respect to r by $'$),

$$\begin{array}{lll} & \Gamma^0_{0r} = \Phi' & \\ \Gamma^r_{00} = e^{2(\Phi-\Lambda)}\Phi' & & \Gamma^r_{rr} = \Lambda' \\ \Gamma^r_{\theta\theta} = -re^{-2\Lambda} & \Gamma^r_{\phi\phi} = -re^{-2\Lambda}\sin^2\theta & \\ \Gamma^\theta_{r\theta} = r^{-1} & \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta & \\ \Gamma^\phi_{r\phi} = r^{-1} & \Gamma^\phi_{\theta\phi} = \cot\theta & \end{array} \tag{131}$$

The Ricci scalar looks like ,

$$R = e^{-2\Lambda}(-2\Phi'' - 2(\Phi')^2 + 2\Phi'\Lambda' - 4r^{-1}\Phi' + 4r^{-1}\Lambda' - 2r^{-2}) + 2r^{-2}\,. \tag{132}$$

The Einstein tensor has the following non-zero components,

$$\begin{aligned}
G_{00} &= \frac{1}{r^2} e^{2\Phi} \frac{d}{dr}[r(1-e^{-2\Lambda})] \\
G_{rr} &= -\frac{1}{r^2} e^{2\Lambda}(1-e^{-2\Lambda}) + \frac{2}{r}\Phi' \\
G_{\theta\theta} &= r^2 e^{-2\Lambda}(\Phi'/r - \Lambda'/r + \Phi'' + (\Phi')^2 - \Phi'\Lambda') \\
G_{\phi\phi} &= \sin^2\theta G_{\theta\theta}
\end{aligned} \tag{133}$$

In the vacuum outside the star, for example, we can set the stress-energy tensor equal to zero. The (00) component gives us,

$$r(1-e^{-2\Lambda}) = \text{constant} = B\,, \tag{134}$$

or equivalently,

$$e^{-2\Lambda} = 1 - \frac{B}{r}\,. \tag{135}$$

The (rr) component gives,

$$2\,d\Phi = \frac{B}{r(r-B)}\,dr = d\left(\ln(1-B/r)\right)\,, \tag{136}$$

so that, apart from a constant,

$$e^{2\Phi} = 1 - \frac{B}{r}\,. \tag{137}$$

We can absorb this constant in our definition of time, and write the metric as,

$$ds^2 = -(1-B/r)dt^2 + \frac{1}{1-B/r}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{138}$$

If we want the field far from an object of mass M to resemble the Newtonian field, then we must have $-g_{00} = 1 + 2\phi/c^2 = 1 - 2GM/(rc^2)$. So that the constant is basically $2GM/c^2$. So, the metric now looks like (at large r),

$$ds^2 \approx -(1-2GM/(rc^2))dt^2 + (1+2GM/(rc^2)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{139}$$

This is the Schwarzschild metric.

5.3 Orbits in Schwarzschild geometry

From the metric we can immediately infer that since it is independent of time, p_0 is constant. Also p_ϕ is a constant. Let us define two constants for particles and photons:

$$\begin{aligned}
\tilde{E} = -p_0/m \quad (particles), &\qquad E = p_0 \quad (photons) \\
\tilde{L} = p_\phi/m, &\qquad L = p_\phi\,,
\end{aligned} \tag{140}$$

where m is the rest mass of the particle. Also it can be shown (we are not going to do it here, but we know it from Newtonian theory anyway) that the orbit is confined in a plane, so that θ is a constant. Now $p_\theta \propto \frac{d\theta}{d\lambda} = 0$ where λ is some parameter on the orbit. So, we have

$$\begin{aligned}
p^0 = g^{00} p_0 &= \frac{m}{1-2M/r}\tilde{E} & \qquad p^0 &= \frac{E}{1-2M/r} \\
p^r &= m\frac{dr}{d\tau} & p^r &= \frac{dr}{d\lambda} \\
p^\phi = g^{\phi\phi} p_\phi &= \frac{m}{r^2}\tilde{L} & p^\phi &= \frac{L}{r^2}
\end{aligned} \tag{141}$$

Now the equation $\vec{p}.\vec{p} = -m^2$ gives us, for particles,

$$\frac{-m^2\tilde{E}^2}{(1-2M/r} + \frac{m^2}{1-2M/r}(\frac{dr}{d\tau})^2 + \frac{m^2\tilde{L}^2}{r^2} = -m^2\,, \tag{142}$$

and for photons,

$$\frac{-E^2}{1-2M/r} + \frac{1}{1-2M/r}(\frac{dr}{d\lambda})^2 + \frac{L^2}{r^2} = 0\,. \tag{143}$$

Or equivalently,

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - (1-2M/r)(1+\tilde{L}^2/r^2) \tag{144}$$

for particles, and for photons,

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - (1-2M/r)(L^2/r^2)\,. \tag{145}$$

We can define the effective potentials as,

$$\tilde{V}^2(r) = (1-2M/r)(1+\tilde{L}^2/r^2) \qquad V^2(r) = (1-2M/r)(L^2/r^2)\,. \tag{146}$$

So that, for particles, we can write $\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(r)$, or,

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{d}{dr}\tilde{V}^2(r)\,. \tag{147}$$

This means that circular orbits ($r =$ constant) is possible at the minimum and maximum values of $\tilde{V}^2(r)$. The unstable point is at,

$$\frac{d}{dr}\left[(1-2M/r)(1+\tilde{L}^2/r^2)\right] = 0\,, \tag{148}$$

which gives,

$$r = \frac{\tilde{L}^2}{2M}\left(1 \pm \sqrt{(1-12M^2/\tilde{L}^2)}\right)\,. \tag{149}$$

So, no stable orbits are possible if $\tilde{L}^2 < 12M^2$. The effective potential in Newtonian theory is different, and is given by (writing $\tilde{l} = l/m = r^2\dot{\phi}$ and omitting G),

$$V_{Newt}(r) = -M/r + \frac{\tilde{l}^2}{2r^2}\,, \tag{150}$$

so that the distance for a circular orbit was given by,

$$r = \frac{\tilde{l}^2}{M}(= \frac{\tilde{l}^2}{GM})\,. \tag{151}$$

For photons there is only a possibility of a unstable orbit and it is at $r = 3M$.

5.4 Perhelion Shift : Optional

If the orbit of a slightly non-circular orbit does not close, its perihelion will shift in time. We can derive the shift (in ϕ) by writing an equation of the orbit in terms of r and ϕ. We can write,

$$\frac{d\phi}{d\tau} = U^{\phi} = \frac{p^{\phi}}{m} = g^{\phi\phi}\frac{p_{\phi}}{m} = g^{\phi\phi}\tilde{L} = \frac{\tilde{L}^2}{r^2}\,. \tag{152}$$

Using the expression for $(dr/d\tau)$ and this expression, one gets,

$$(\frac{dr}{d\phi})^2 = \frac{\tilde{E}^2 - (1-2M/r)(1+\tilde{L}^2/r^2)}{\tilde{L}^2/r^4}\,. \tag{153}$$

We define $r = 1/u$ (that is, $dr = -du/u^2$) and write,

$$\begin{aligned}(\frac{du}{d\phi})^2 &= \frac{\tilde{E}^2 - (1-2M/r)(1+\tilde{L}^2/r^2)}{\tilde{L}^2/r^4} \\ &= \frac{\tilde{E}^2}{\tilde{L}^2} - (1-2Mu)(\frac{1}{\tilde{L}^2}+u^2)\,.\end{aligned} \tag{154}$$

We recover the Newtonian approximation by neglecting the u^3 terms, which gives,

$$(\frac{du}{d\phi})^2_{Newt} = \frac{\tilde{E}^2}{\tilde{L}^2} - \frac{1}{\tilde{L}^2}(1-2Mu) - u^2\,. \tag{155}$$

The circular orbit is given by (as discussed in the earlier section) $u = M/\tilde{L}^2$. Let us definite a parameter $y = u - M/\tilde{L}^2$ to describe the deviation from circularity. Then one can write in the Newtonian case,

$$(\frac{dy}{d\phi})^2_{Newt} = \frac{\tilde{E}^2-1}{\tilde{L}^2} + M^2/\tilde{L}^4 - y^2\,, \tag{156}$$

whose solution is of the type $y = A\cos(\phi+B)$ where A and B are constants. This gives for the orbit,

$$\frac{1}{r} = M/\tilde{L}^2 + A\cos(\phi+B)\,, \tag{157}$$

which is the equation for an ellipse. The particle comes back to the same r after it goes through $\Delta\phi = 2\pi$.

Let us do the same exercise keeping the u^3 term. We can write for the same parameter y,

$$\begin{aligned}(\frac{dy}{d\phi})^2 &= \frac{\tilde{E}^2}{\tilde{L}^2} - (1-2M(y+M/\tilde{L}^2))\left(\frac{1}{\tilde{L}^2}+y^2+\frac{M^2}{\tilde{L}^4}+2y\frac{M}{\tilde{L}^2}\right) \\ &= \frac{\tilde{E}^2+M^2/\tilde{L}^2-1}{\tilde{L}^2} + \frac{2M^4}{\tilde{L}^6} + \frac{6M^3}{\tilde{L}^2}y + (\frac{6M^2}{\tilde{L}^2}-1)y^2 \\ &= A^2k^2 + ky_0y - k^2y^2\,.\end{aligned} \tag{158}$$

Here $k = \sqrt{(1-6M^2/\tilde{L}^2)}$. The solution for this is,

$$y = y_0 + A\cos(k\phi+B)\,, \tag{159}$$

which shows that the orbit would come back to the same r only after $k\phi$ has gone through an angle 2π. For each orbit though the shift would be

$$\Delta\phi = 2\pi/k = 2\pi(1-6M^2/\tilde{L}^2)^{-1/2} \approx 2\pi(1+3M^2/\tilde{L}^2)\,, \tag{160}$$

so that the ‘shift’ is $6\pi M^2/\tilde{L}^2$ per orbit. Now, for circular orbits, approximately, $\tilde{L}^2 \approx Mr$. So that the shift is $6\pi M/r (\equiv 6\pi GM/(c^2 r))$. For Mercury ($r = 5.6 \times 10^7$ km) and $M_\odot = 2 \times 10^{33}$ g, one has a shift of 5×10^{-7} radians per orbit. Since each orbit takes 0.24 year, this comes to about $43''$ per century.

Note that a perihelion shift is also there in the Newtonian theory, when one takes into account the fact that there are other massive bodies in the solar system. The shift derived above is the difference between the observed value and what is predicted from the Newtonian theory.

5.5 Bending of light : optional

Using the expressions for $(dr/d\lambda)^2$ and $(d\phi/d\lambda)^2$ for photons, one can write,

$$(\frac{d\phi}{dr})^2 = \frac{L^2/r^4}{(E^2-(1-2M/r)(L^2/r^2))}\,, \tag{161}$$

so that,

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2\left[\frac{E^2}{L^2}-(1-2M/r)\frac{1}{r^2}\right]}\,. \tag{162}$$

We define $b = L/E$ and $u = 1/r$ $(dr = -r^2du)$, and write,

$$\frac{d\phi}{du} = \frac{1}{\left[\frac{1}{b^2}-u^2(1-2Mu)\right]^{1/2}}\,. \tag{163}$$

In Newtonian theory, one would have,

$$\frac{d\phi}{du} = \frac{1}{\left[\frac{1}{b^2}-u^2\right]^{1/2}}\,, \tag{164}$$

whose solution would be the straight line,

$$r\sin(\phi-\phi_0) = b\,, \tag{165}$$

which is the orbit of the photon with an impact parameter b (the minimum distance). We define $y = u(1-Mu)$ so that $u = y/(1-Mu) \approx y(1+Mu) \approx y(1+My)$, and $du = (1+2My)dy$. We then have,

$$\frac{d\phi}{dy} = \frac{1+2Mu}{\left[\frac{1}{b^2}-u^2\right]^{1/2}}\,, \tag{166}$$

whose solution is,

$$\phi = \phi + 2M/b + \arcsin(by) - 2M\sqrt{\frac{1}{b^2}-y^2}\,. \tag{167}$$

At infinity, one has $y = 0$ and $\phi = \phi_0$. At the closest approach, $r = b$ one has $y \sim 1/b$ and $\phi = \phi_0 + 2M/b + \pi/2$. So, by symmetry, when the photon again recedes to infinity, it would acquire another $\phi = 2M/b + \pi/2$, so that finally, one would have a deviation of $\pi + 4M/b$. So, the bending angle, would be $4M/b(\equiv 4GM/(c^2b))$. For Sun, at the solar radius, this amounts to $\sim 1''.74$.

5.6 Gravitational redshift

It is clear that any relativistic theory of gravity would predict redshift due to gravitation. Consider the following gedanken experiment. There is an endless chain running between the Earth and the Sun carrying buckets containing atoms in an excited state on one side and an equal number of atoms in the ground state on the other side. Since the excited atoms possess greater energy and so are heavier and so that side of the chain would fall toward the Sun whose gravitational field dominates. Suppose we have a device which returns an atom to the ground state, collects the emitted photon and reflects it back to the Earth where it is used to excite an incoming atom in the ground state. We would then have a perpetual motion machine. So, something must be wrong with the argument somewhere. It is because the radiation coming to Earth is not sufficiently energetic to excite the incoming ground state atom. In other words, it loses energy while climbing up the gravitational field of the Sun.

let us quantify this redshift. We consider two observers carrying ideal atomic clocks whose worldlines are $x^\alpha = x_1^\alpha$ and $x^\alpha = x_2^\alpha$. Let the first observer send out radiation to the second observer. We denote the time separation between successive wave crests as measured by the first clock by $d\tau(1)$ in terms of proper time and by dx_1^0 in terms of coordinate time. It follows that $d\tau(1)^2 = g_{00}(x_1^\alpha)\,(dx^0)_1)^2$. Let the corresponding interval of reception recorded by the second observer be $d\tau(2)$ in proper time, which is given by, $d\tau(2)^2 = g_{00}(x_2^\alpha)\,(dx^0)_2)^2$. We assume that the space-time is static, that is $dx_1^0 = dx_2^0$, because otherwise there would be a build-up or depletion of wave crests between the two observers, in violation of the static assumption. So, the ratio of the proper times is $\sqrt{g_{00}(x_2^\alpha)/g_{00}(x_1^\alpha)}$. This tells one how many times the second clock has ticked between the reception of two wave crests. It follows that if the atomic clock has characteristic frequency ν_{em}, then the second observer will measure a frequency $\nu_{obs} = \nu_{em}\sqrt{g_{00}(x_1^\alpha)/g_{00}(x_2^\alpha)}$.

In the weak field limit $g_{00} = -1-2\phi$, and one has for the shift in energy and frequency,

$$\frac{\Delta\nu}{\nu} = \frac{\nu_{em}-\nu_{obs}}{\nu_{em}} = 1 - \frac{1+\phi_1}{1+\phi_2} \sim \phi_2 - \phi_1 (\equiv (\phi_2-\phi_1)/c^2)\,. \tag{168}$$

For the Schwarzschild metric one has,

$$\frac{\Delta\nu}{\nu} \sim -\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\,. \tag{169}$$

Numerically, the shift for a height of 100 ft, say, in the Earth's graviational field, is only of order 10^{-15}. The experiment by Pound and Rebka in 1960 showed that it was true within an accuracy of 1 %.

5.7 Friedmann-Robertson-Walker metric

Let us derive a metric which will be useful to us for cosmology. We will soon see that observations tell us that our universe at large scale is homogeneous and isotropic. To be precise, we assume that spacetime can be sliced into (hyper)surfaces of constant time which are homogeneous and isotropic. We will also assume that the mean rest frame of galaxies agrees with this definition of simultaneity.

Each galaxy is thought to have no random motions, and the time coordinate t is assumed to be the proper time for each galaxy. We will however allow for time dependence of the metric coefficients, since we will soon see that our universe does seem to expand. So, at a given time t_0, the constant-time surface has a line element,

$$dl^2(t_0) = h_{ij}dx^i dx^j\,. \tag{170}$$

Then the expansion of the constant-time surface can be represented by

$$dl^2(t_1) = f(t_1,t_0)h_{ij}(t_0)dx^i dx^j = h_{ij}(t_1)dx^i dx^j\,, \tag{171}$$

where we have assumed that all h_{ij}s increase at the same rate, to make the expansion isotropic. In general, then we can write,

$$dl^2(t) = R^2(t) h_{ij} dx^i dx^j \,, \tag{172}$$

where R is an overall scale factor, which equals 1 at t_0. So, the metric would be,

$$ds^2 = -dt^2 + g_{0i} dt\, dx^i + R^2(t) h_{ij} dx^i dx^j \,, \tag{173}$$

where we have put $g_{00} = -1$ since we assumed that t is the proper time along a line $dx^i = 0$. But if the local Lorentz frame of a galaxy has to agree with the definition of simultaneity given by $t =$ constant, then in the comoving frame the base vectors $\vec{e}_0$ and $\vec{e}_i$ should be orthogonal. This means that $g_{0i} = \vec{e}_0 . \vec{e}_i = 0$. So that we have,

$$ds^2 = -dt^2 + R^2(t) h_{ij} dx^i dx^j \,. \tag{174}$$

Now, since the constant time hypersurface is isotropic about every point, it must be spherically symmetric about the origin of the coordinates. We have seen that such a metric has the line element,

$$dl^2 = e^{2\Lambda(r)} dr^2 + r^2\, d\Omega^2 \,. \tag{175}$$

Also, isotropy about every point implies homogeneity. In particular, this means that the Ricci scalar should have the same value everywhere. We can use the expression for the Ricci scalar we had derived for centrally symmetric metrics, after putting $\Phi = 0$ as,

$$R = e^{-2\Lambda}(4\Lambda'/r - 2/r^2) + 2/r^2 = \frac{2}{r^2}\frac{d}{dr}(r - re^{-2\Lambda}) = k \,, \tag{176}$$

where k is a constant. This is easily integrated to give,

$$e^{2\Lambda} = g_{rr} = \frac{1}{1 - \frac{1}{6}kr^2 - A/r} \,, \tag{177}$$

where A is a constant of integration. The assumption of local flatness at $r = 0$ makes $A = 0$. Redefining the constant k, we then have,

$$ds^2 = -dt^2 + R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] . \tag{178}$$

This is the Friedmann-Robertson-Walker metric. We can define the coordinate r in such a way that k takes only three values $-1, 0, 1$. For example, if $k = -3$, then we can redefine $\overline{r} = \sqrt{3}r$, $\overline{k} = -1$, and $\overline{R} = 1/\sqrt{3}R$, and the line element becomes,

$$dl^2 = \overline{R}^2(t) \left[\frac{d\overline{r}^2}{1 - \overline{k}\overline{r}^2} + \overline{r}^2 d\Omega^2 \right] . \tag{179}$$

One cannot however change the sign of k.