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Lecture notes for structure formation: Biman B. Nath

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1 Structure formation

1.1 Preliminaries: suppression and growth of inhomogeneities

We will assume that we can isolate a region small enough for the Newtonian potential energy and the relative particle velocities to be small, so that we can use the Newtonian mechanics. If one considers a homogeneous mass distribution, the gravitational potential energy belonging to the mass M contained in the sphere of proper radius R is,

$$\Phi \sim \frac{GM}{R} \sim G\rho_b R^2 \sim (HR)^2\,, \tag{1}$$

where we have assumed $\Omega_0 \sim 1$ for the last equality, so that the expansion is dominated by the mass term, which is true within an order of magnitude at present epoch. If the size R of the structures considered is small compared to the Hubble radius H^{-1}, the Hubble velocities are non-relativistic. If the density contrast is small, this also means that the gravitational potential is non-relativistic. And the structures we want to study, like the formation of galaxies, are quite non-relativistic (apart from things happening in the nucleus).

First we will consider matter as a pressureless fluid. The fluid equations are (mass conservation, equation of motion and Poisson equation), are,

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \nabla.(\rho\vec{v}) = 0\,, \\ &\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\nabla\phi\,, \\ &\nabla^2\phi = 4\pi G\rho\,. \end{aligned} \tag{2}$$

This is in the Eulerian coordinate system, which includes the expansion of the universe. Another approach in fluid dynamics is to use the Lagrangian coordinates, which do not change in time. Here we will instead use a similar coordinate system, called the comoving coordinate system (which we have already encountered). We define a comoving coordinate system $\vec{x}$ so that points expanding with the background universe have fixed coordinates in the comoving frame. The comoving coordinates $\vec{x}$ are related to the proper (physical) coordinates $\vec{r}$ by $\vec{r} = R(t)\vec{x}$. The proper velocity at any point is $\vec{v} = \dot{R}\vec{x} + R\dot{\vec{x}}$. The second term is the peculiar velocity and it describes departures from uniform Hubble flow $\vec{v} = \dot{R}\vec{x}$. We will define a peculiar velocity in the comoving frame $\vec{u} \equiv \dot{\vec{x}}$. We also assume that the density can be written as,

$$\rho(\vec{x},t) = \overline{\rho}(t)(1+\delta(\vec{x},t)), \tag{3}$$

so that we can write the fluid equations in the coordinate frame as,

$$\begin{aligned}
&\frac{\partial\delta}{\partial t} + \nabla_x.\vec{u} + \nabla_x.(\vec{u}\delta) = 0, \\
&\frac{\partial\vec{u}}{\partial t} + 2\frac{\dot{R}}{R}\vec{u} + (\vec{u}.\nabla_x)\vec{u} = -\nabla_x\tilde{\phi}/R^2, \\
&\nabla_x^2\tilde{\phi}/R^2 = 4\pi G\overline{\rho}\delta,
\end{aligned} \tag{4}$$

where ∇_x denotes gradients with respect to the comoving coordinates x (that is, $\nabla_x/R = \nabla$). Note that we have not yet linearlized the set of equations. Also, note that these equations are valid in the comoving frame in which Hubble expansion ($\dot{R}\vec{x}$) has been taken out (it is constant in this frame). This meant that the term $(\dot{R}\vec{x}.\nabla)(R\vec{u}) = \dot{R}u$.

If we Fourier transform these equations, assuming that the perturbed quantities (the fractional density perturbation δ, the three components of peculiar velocity $\vec{u}$ and the potential perturbation $\tilde{\phi}$) are periodic in a large box of volume V, that is,

$$\begin{aligned}
\delta(\vec{x},t) &= \frac{(2\pi)^{3/2}}{V^{1/2}}\sum_{\vec{k}}\delta_{\vec{k}}e^{i\vec{k}x}, \\
\delta_{\vec{k}} &= \frac{(2\pi)^{3/2}}{V^{1/2}}\int\delta e^{-i\vec{k}x}d^3x,
\end{aligned} \tag{5}$$

where $\vec{k}$ is a comoving wavenumber. Note that $\delta(\vec{x})$ is dimensionless but δ_k has units of length$^{3/2}$. We then get,

$$\begin{aligned}
&\frac{d\delta_{\vec{k}}}{dt} + i\vec{k}.\vec{u}_{\vec{k}} + \sum_{\vec{k}'} i\delta_{\vec{k}'}(\vec{k}.\vec{u}_{\vec{k}-\vec{k}'}) = 0, \\
&\frac{d\vec{u}_{\vec{k}'}}{dt} + 2\frac{\dot{R}}{R}\vec{u}_{\vec{k}'} + \sum_{\vec{k}'} i[\vec{u}_{\vec{k}'}.(\vec{k}-\vec{k}')]\vec{u}_{\vec{k}-\vec{k}'} = i\frac{\vec{k}}{R^2}\tilde{\phi}_{\vec{k}}, \\
&\tilde{\phi}_{\vec{k}}/R^2 = -4\pi G\overline{\rho}\frac{\delta_{\vec{k}}}{|\vec{k}|^2}.
\end{aligned} \tag{6}$$

The terms under the summation sign denote coupling between the different Fourier modes, which we will neglect in the linear perturbation theory.

Now, neglecting the non-linear terms, we have a single second-order differential equation,

$$\frac{d^2\delta_{\vec{k}}}{dt^2} + 2\frac{\dot{R}}{R}\frac{d\delta_{\vec{k}}}{dt} - 4\pi G\overline{\rho}\delta_{\vec{k}} = 0. \tag{7}$$

In a matter dominated $\Omega_0 = 1$ universe, $R(t) \propto t^{2/3}$ and so we have,

$$\frac{d^2\delta_{\vec{k}}}{dt^2} + \frac{4}{3t}\frac{d\delta_{\vec{k}}}{dt} - \frac{2}{3t^2}\delta_{\vec{k}} = 0. \tag{8}$$

So, one has two solutions with a growing and decaying mode and the general solution is,

$$\delta_{\vec{k}} = A_{\vec{k}}t^{2/3} + B_{\vec{k}}t^{-1}. \tag{9}$$

So, the growing mode has an amplitude δ that is proportional to the scale factor, and so an 0.01 % perturbation at $1 = z = 1000$ will become a 10% perturbation by $z = 0$. In a matter dominated $\Omega \ll 1$ universe, $R(t) \propto t$ at $z < 1/\Omega_0 - 1$, and the solution is,

$$\delta_{\vec{k}} = A_{\vec{k}} + B_{\vec{k}} t^{-1}\,, \tag{10}$$

so that the fluctuations freeze out at $z \sim 1/\Omega_0 - 1$ when free expansions start. If there is a cosmological constant then if it dominates the density, then one has,

$$\frac{d^2\delta_{\vec{k}}}{dt^2} + 2H\frac{d\delta_{\vec{k}}}{dt} = 0\,, \tag{11}$$

which has solutions $\delta =$ constant and $\exp(-2Ht)$. Thus the growing mode stops growing when the universe becomes vacuum- dominated.

1.2 Fluctuation in a smooth relativistic background

Because of free-streaming of collisionless relativistic particles, fluctuations in a sea of relativistic particles would grow differently. In this case it is safe to assume that the relativistic particles would be smoothly distributed in scales much smaller the Hubble scale, and so the perturbation in the non-relativistic matter would provide the gravitational source term,

$$\frac{d^2\delta_{\vec{k}}}{dt^2} + 2\frac{\dot{R}}{R}\frac{d\delta_{\vec{k}}}{dt} - 4\pi G\overline{\rho_m}\delta_{\vec{k}} = 0\,, \tag{12}$$

and the only effect of the relativistic component is to change the background expansion rate,

$$(\dot{R}/R)^2 = (8/3)\pi G(\overline{\rho_m} + \overline{\rho_R}\,. \tag{13}$$

Here, $\overline{\rho_m} \propto R^{-3}$ and $\overline{\rho_R} \propto R^{-4}$. If we transfer the variable t to $\eta = \overline{\rho_m}/\overline{\rho_R}$, then we would have,

$$\frac{d^2\delta_{\vec{k}}}{d\eta^2} + \frac{(2+3\eta)}{2\eta(1+\eta)}\frac{d\delta_{\vec{k}}}{d\eta} = \frac{3}{2}\frac{\delta_{\vec{k}}}{\eta(1+\eta)}\,. \tag{14}$$

This has a growing mode solution of $\delta \propto 1 + (3/2)\eta$ and a decaying mode solution. So these fluctuations cannot grow until $\eta > 1$.

So linear perturbations do not grow if $\eta \ll 1$ that is before the matter-radiation equality. Basically the universe expands so fast then that matter has little chance to collapse. This is also known as the Mészáros effect (Mészáros 1974).

1.3 Scales larger than c/H

Crudely speaking, one can treat the *very* large scale perturbations as separate homogeneous universes. We can use the evolution of $(\Omega^{-1} - 1)$ that we derived earlier for this case. We have that, $(\Omega^{-1} - 1) \propto (1/\rho R^2)$ which is proportional to $(1+z)^{-1}$ for matter dominated and to $(1+z)^{-2}$ for radiation dominated case. So, δ which is proportional to $(\Omega^{-1} - 1)$ grows as $\delta \propto R$ for matter dominated and $\delta \propto R^2$ for radiation dominated case.

1.4 Zel'dovich Approximation

It is reasonable to assume that only the growing mode is present with some significant amplitude at present time, since as we will see later that the fluctuations at the epoch of recombination were small. Then equation (**??**) can be rewritten (after summing over all Fourier modes)

$$\delta(\vec{x},t) = D(t)\delta_0(\vec{x}) \,, \tag{15}$$

where D denotes the growing mode (for $\Omega = 1$, $D(t) \propto R \propto t^{2/3}$ and so on) and δ_0 some initial perturbation. So, the density field self-similarly in time. It turns out that the gravitational acceleration and the peculiar velocity also evolves self-similarly. If we substitute the above equation into the Poisson equation, we have,

$$\phi(\vec{x},t) = \frac{D}{R}\phi_0(\vec{x}) \,, \tag{16}$$

where

$$\nabla^2\phi_0 = 4\pi G(\overline{\rho}R^3)\delta_0(\vec{x}) \,. \tag{17}$$

Now, the linearlized form of Euler's equation (neglecting the u^2 terms) is, $d\vec{u}/dt + 2(\dot{R}/R)\vec{u} = -\nabla_x\phi/R^2$, which can be written as $d(R^2\vec{u})/dt = -\nabla_x\phi = -(D/R)\nabla_x\phi_0$. This can be integrated to give,

$$\vec{u} = -\Big(\frac{1}{R^2}\int\frac{D}{R}dt\Big)\nabla_x\phi_0 \,. \tag{18}$$

Since D solves the equation (7), we have $d(R^2\dot{D})/dt = 4\pi G\overline{\rho}DR^2 = 4\pi G\overline{\rho_{co}}D/R$, where we have written $\rho = \rho_{co}/R^3$. This means that $R^2\dot{D} = 4\pi G\overline{\rho_{co}}\int\frac{D}{R}dt$, so that we can write for the peculiar velocity,

$$\vec{u} = -\frac{\dot{D}}{4\pi G\overline{\rho_{co}}}\nabla_x\phi_0 \,, \tag{19}$$

which on integration gives,

$$\vec{x} = \vec{x}_0 - \frac{D(t)}{4\pi G\overline{\rho}R^3}\nabla_x\phi_0 \,. \tag{20}$$

This formulation of linear theory is due to Zel'dovich (1970, A&A, 5, 84), which specifies the growth of structure by giving the displacement $\vec{x}-\vec{x}_0$ and the peculiar velocity $\vec{u}$ of each mass element in terms of its initial position $\vec{x}_0$. Therefore it is a Langrangian description.

To appreciate what it means, let us write it in a slightly different form, (in one dimension)

$$x(t) = x_0 + D(t)\,f(x_0) \,, \tag{21}$$

which shows that the mass elements essentially go with the Hubble flow with some perturbations. The motion is like inertial motion and the distance travelled is proportional to $D(t)$ and the initial 'kick' $f(x_0)$. At $t > 0$, the density can be found by using the mass conservation law $\rho(x,t)dx = \rho_0 dx_0$. One can find that the density, as a function of the initial Lagrangian coordinate x_0, is

$$\rho(x_0,t) = \frac{\rho_0}{1-D(t)\alpha} \,, \tag{22}$$

where $\alpha = -df(x_0)/dx_0$. (In three dimension, this would mean taking the Jacobian instead of the simple dx/dx_0 and so $df(x_0)/dx_0$ would be a tensor in general, called the deformation tensors. If this tensor is symmetrical, then one can find a coordinate system in which it is diagonal, and so there would be three parameters like α here.) This shows that at some time when $D = 1/\alpha$ there would be infinite, and so this approximation in general predicts caustics. In 3 dimension, the deformation is largest in the direction for which α is most negative, and so one expects flattened structures, which were christened 'pancakes' by Zel'dovich.

As a matter of fact in 1 D, as long as different sheets of matter do not cross, Gauss's theorem shows that,

$$\vec{g} = -\frac{1}{R}\nabla_x\phi = -4\pi G\bar{\rho}(Rx_0 - Rx)\,, \tag{23}$$

which means that $x = x_0 - \frac{1}{4\pi G\bar{\rho}R^2}\nabla_x\phi = x_0 - \frac{D(t)}{4\pi G\bar{\rho}R^3}\nabla_x\phi_0$, which is equivalent to the previous expression.

Zel'dovich proposed using this approximation until the trajectories cross, and this has been shown to be an excellent approximation by numerical simulations. This is used extensively to set up initial conditions for numerical simulations.

There is an interesting similarity between Zel'dovich approximation and geometric optics, which leads to a similiarity between the pattern of light seen at the bottom of a swimming pool and that of the large-scale structure in the universe. Consider a horizontal, transparent plate illuminated from below (in the swimming pool, it is illuminated by reflected light) by parallel rays. The plate has a flat base at the plane $r = 0$ and a smoothly varying thickness which is specified by thickness $h = h(x,y)$. When the rays pass through such a plate they are deflected differently at different points. If we denote the deflection angle by s and if it is small then the 2D coordinates of the ray entering the plate at the point with coordinate $\vec{q} = (q_1, q_2)$ depend on z as,

$$\vec{r}(z,\vec{q}) = \vec{q} + z\vec{s}(\vec{q})\,, \tag{24}$$

where $s_i(\vec{q}) = -(n-1)\frac{\partial h(\vec{q})}{\partial q_i}$, n being the refractive index, assumed here to be independent of the wavelength. This is very similar to the expression for Zel'dovich approximation.

1.5 Jeans length

If we now include pressure, the continuity equation would read,

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\nabla\phi - \frac{1}{\rho}\nabla p\,. \tag{25}$$

The adiabatic sound speed is $c_s^2 = (\partial p/\partial\rho)_S$, so that including pressure, the final second-order equation would read,

$$\frac{d^2\delta_{\vec{k}}}{dt^2} + 2\frac{\dot{R}}{R}\frac{d\delta_{\vec{k}}}{dt} = \left[4\pi G\bar{\rho} - (c_s k/R)^2\right]\delta_{\vec{k}}\,. \tag{26}$$

One defines the Jeans length as $\lambda_J = 2\pi R/k_J = c_s\sqrt{\pi/G\rho_m}$. So if the scale size of a perturbation is larger than the Jean's length, pressure gradients can be ignored and the matter behaves like a pressure-less fluid as we have seen earlier. But if $\lambda \ll \lambda_J$ then gravity can be ignored and the perturbations would oscillate like acoustic waves.

In the standard cosmology, the universe recombines around $z \sim 1000$. After the recombination, the relevant sound speed is the adiabatic sound speed of a monoatomic gas, since photons are no longer coupled to matter. So, $c_s^2 = \sqrt{(5kT/3m_p)}$, and so at $z \sim 1000$ ($T \sim 3000$ K), we have $\lambda_J \sim 3 \times 10^{19} (\Omega h^2)^{-1/2}$ cm. The corresponding mass Jeans mass is $M_J \sim (4/3\pi)\rho_m \lambda_J^3 \sim 1.3 \times 10^6 (\Omega h^2)^{-1/2}$ $M_\odot$. Notice that this mass is close to that of globular clusters and it has been suggested that globular clusters are the first objects to collapse after recombination, although there are other possible astrophysical origins of globular clusters.

Before the recombination, photons are tightly coupled to matter through Thomson scattering and the adiabatic sound speed is $c_s = (c/\sqrt{3})((3/4)(\rho_m/\rho_\gamma)+1)^{-1/2}$. (This follows from the fact that $c_s^2 = \frac{4}{3}\frac{p}{\rho}$, where $p \sim \frac{1}{3}\rho c^2$ and $\rho = \rho_m + \rho_\gamma$.) As $\rho_\gamma/\rho_m \sim (z/42000)(\Omega h^2)^{-1}$, so that just before recombination, one has $c_s \sim (c/10)(\Omega h^2)^{-1}$ and the Jeans mass is $M_J \sim 9 \times 10^{16} (\Omega h^2)^{-2}$ $M_\odot$. So before decoupling, perturbations on scales smaller than supercluster scales oscillate like sound waves.

1.6 Silk damping

Although photons and electrons are tightly coupled before recombination, the coupling is not perfect and this leads to damping of perturbations, which is called the Silk damping. Instead of solving the Boltzmann's equation, we will simply estimate the Silk damping scale. Since the damping is caused by free-streaming of photons out of the overdense regions, we need to calculate the mean free path of photons, which is,

$$\lambda_\gamma \equiv \frac{1}{X_e n_e \sigma_T} \sim 1.3 \times 10^{29} X_e^{-1} R^3 (\Omega_B h^2)^{-1}\,\mathrm{cm}\,, \tag{27}$$

where X_e is the ionization fraction. For perturbations of size $\lambda \lesssim \lambda_\gamma$, the perfect fluid approximation breaks down, and photon streaming would damp such perturbation. Now, in a time Δt a photon suffers $N = \Delta t/\lambda_\gamma(t)$ collisions and so undergoes a random walk characterized by a mean coordinate distance Δr where,

$$(\Delta r)^2 \approx N \frac{\lambda_\gamma(t)^2}{R(t)^2} \approx \frac{\Delta t}{\lambda_\gamma(t)} \frac{\lambda_\gamma(t)^2}{R(t)^2}\,. \tag{28}$$

The total coordinate distance travelled until the time of decoupling is,

$$\lambda_S^2 = \int_0^{t_{dec}} dt \frac{\lambda_\gamma}{R^2(t)} = \frac{3}{5} \frac{t_{dec}\lambda_\gamma(t_{dec})}{R_{dec}^2}\,, \tag{29}$$

since $R \propto r^{2/3}$ during the matter dominated decoupling era. Using a value of $X_e \sim 0.1$ around $(1+z_{dec}) = R_{dec}^{-1} = 1100$, one has,

$$\begin{aligned} \lambda_S &= 3.5(\Omega_0/\Omega_B)^{1/2}(\Omega_0 h^2)^{-3/4}\,\mathrm{Mpc} \\ M_S &= 6.2 \times 10^{12}(\Omega_0/\Omega_B)^{3/2}(\Omega_0 h^2)^{-5/4}\,M_\odot\,. \end{aligned} \tag{30}$$

This scale is close to that of clusters.

If there are nearly collisionless components in matter there would be free streaming of this components and that would also damp perturbations. Once a species decouples from plasma, it travels in free fall in the expanding universe. If we choose the motion of the particle to be radial then the motion of the particle is $R(t)dr = v(t)dt$. We are interested in free-streaming just before the growth of perturbations take place, that is before t_{eq}. The comoving free-streaming scale just before the matter radiation equality is,

$$\lambda_{FS} = \int_0^{t_{eq}} \frac{v(t')}{R(t')} dt' = \int_0^{t_{NR}} \frac{c}{R(t')} dt' + \int_{t_{NR}}^{t_{eq}} \frac{v(t')}{R(t')} dt' \,, \tag{31}$$

where we have split the integral into two, when in the relativistic regime $v \sim c$ and the non-relativistic regime. Since the particle is freely propagating, $p = mv \propto R^{-1}$ and so $v \propto R^{-1}$ in the non-relativistic regime. So,

$$\lambda_{FS} \approx 2c \frac{t_{NR}}{R_{NR}} + \int_{t_{NR}}^{t_{eq}} \frac{cR_{NR}}{R^2(t')} dt' \,, \tag{32}$$

where we assumed that the universe is radiation dominated when the particle in consideration is relativistic ($t_{NR} < t_{eq}$), which is true for most cases. In the radiation dominated era, $t = t_{NR}(R/R_{NR})^2$ and so,

$$\lambda_{FS} = (ct_{NR}/R_{NR})(2 + \ln(t_{eq}/t_{NR}))\,. \tag{33}$$

A particle X becomes non-relativistic when $k_B T_X \approx m_X c^2/3$, and for weakly interacting particles T_X is likely to be smaller than T, the photon temperature. So, one has, $R_{NR} \sim 7 \times 10^{-7}(keV/m_X)(T_X/T)$ and so, $t_{NR} \sim 1.2 \times 10^7 (keV/m_X)^2 (T_x/T)^2$ sec and $t_{eq}/t_{NR} \sim (m_x/(17(\Omega_0 h^2)(T_X/T)eV))^2$, and finally,

$$\lambda_{FS} \approx 0.2\,\mathrm{Mpc}(m_X/keV)^{-1}(T_X/R)(2 + \ln(t_{eq}/t_{NR})\,. \tag{34}$$

Now, for a two component fermion species we have,

$$\Omega_X = \Big(\frac{mn_x}{\rho_c}\Big)_0 = \Big(\frac{mn_\gamma}{\rho_c}\Big)_0 \frac{n_x}{n_\gamma} = 30\Big(\frac{m_x}{1\,keV}\Big)\frac{n_x}{n_\gamma} h^{-2} = 30\Big(\frac{m_x}{1\,keV}\Big)\Big(\frac{T_x}{T}\Big)^3 h^{-2}. \tag{35}$$

So we can write,

$$\lambda_{FS} \sim 30(\Omega_x h^2)^{-1}(T_x/T)^4\,\mathrm{Mpc}\,, \tag{36}$$

where we have put $t_{eq}/t_{NR} = 3$.

For a light neutrino, $T_\nu/T \sim 0.7$ and so,

$$\lambda_{FS,\nu} \sim 20\,\mathrm{Mpc}(m_\nu/30eV)^{-1}\,, \tag{37}$$

corresponding to a mass of $4 \times 10^{14}(m_\nu/30eV)^{-2}\,\mathrm{M}_\odot$ (since, $M(\lambda) \sim 1.5 \times 10^{11} M_\odot(\Omega_0 h^2)\lambda^3_{Mpc}$ and $\Omega_\nu h^2 \sim m_\nu/91eV$. More accurate calculations would have given us something like 40 Mpc.

The free-streaming mass can be written in terms of fundamental quantities as $M_{FS,\nu} \approx m^3_{PL}/m^2_\nu$, which happens to be the same form as for the Chandrasekhar mass, with $m_N \to m_\nu$!

1.7 The processed final spectrum

We are in a position now to discuss the final spectrum of matter fluctuation in the universe, after evolving the initial perturbations. We will first consider perturbations in dark matter, and then consider the fate of the baryons.

Let $\delta_k(t_i)$ be the amplitude of perturbation corresponding to some wavenumber $k \propto \lambda^{-1}$ and a mass $M \propto \lambda^3 \propto k^{-3}$ at some initial instant t_i. We would like to find out the value of δ_k at some later time as they cross the horizon and grow in time. Firstly, free streaming will wash out perturbations at scales smaller than λ_{FS}. Then we consider perturbations of comoving wavelength $\lambda_{FS} < \lambda < \lambda_{eq}$ where λ_{eq} is the horizon size at z_{eq}. These modes have scales which are equal to c/H before z_{eq} and do not grow until z_{eq}. Thus for these perturbations $\delta(z) = \delta(z_{eq})(1+z_{eq}) \sim \delta(z_{cross})(1+z_{eq})$. Perturbations with $\lambda > \lambda_{eq}$ ($z_{cross} < z_{eq}$) have $\delta(z) = \delta(z_{cross})(1+z_{cross}) = \delta(z_{cross})(1+z_{cross}/1+z_{eq})(1+z_{eq})$. Here, $R_{cross}\lambda(\propto \lambda t_{cross}^{2/3}) = ct_{cross}$, so $t_{cross} \propto \lambda^3$. Also, since $(R_{eq}/R_{cross}) = (t_{eq}/t_{cross})^{2/3}$, one has $(R_{eq}/R_{cross} = (\lambda_{cross}/\lambda)^2$. So, finally,

$$\delta(z) = \begin{cases} 0 & (\lambda < \lambda_{FS}) \\ \delta(z_{cross})(1+z_{eq}) & (\lambda_{FS} < \lambda < \lambda_{eq}) \\ \delta(z_{cross}(1+z_{eq})(\lambda_{eq}/\lambda)^2 & (\lambda_{eq} < \lambda) \end{cases} \tag{38}$$

So, for perturbations with $\lambda < \lambda_{eq}$ ($z_{cross} > z_{eq}$) lose out on growth, and the final perturbation spectrum has a break at a scale given by k_{break} where $z_{cross} = z_{eq}$.

We can evaluate z_{cross} as a function of the scale $1/k$ using,

$$\frac{R(t)}{k} = \frac{1}{(1+z_{cross})k} = ct = \frac{c}{(1+z)^2}\sqrt{\frac{3}{32\pi G\rho_{r0}}}, \tag{39}$$

where we have used the radiation dominated formula and where ρ_{r0} is the radiation density at present epoch and is well known from CMBR measurements. This means that,

$$k_{break} \propto \frac{\rho_{m0}}{\rho r0} c^{-1}\sqrt{\frac{32\pi G\rho_{r0}}{3}} \propto \Omega_{m0}h^2. \tag{40}$$

We will discuss the fluctuations at different scales in terms of the power spectrum $P(k) = |\delta_k|^2$. We therefore find that there is going to be a first break of the spectrum at λ_{FS}, which depends on whether or not particles like neutrinos dominate the dark matter. If they do, then the spectrum has a sharp peak around λ_{FS}. Other than that, there is a mild break at λ_{eq}. At a given time, the last line of the equation (38) shows that, as the wavelength decreases, δ increases. This happens till $\lambda = \lambda_{eq}$. Below this scale, δ is a constant, as these correspond to scales which crossed the horizon before t_{eq} and did not grow much. In reality, there is some growth, and δ increases only slightly as λ is decreased further.

Suppose the amplitude at a given time t, $\delta_k(z) \propto k^{n/2}$. Since outside the horizon, the perturbations grow as $R^2 \propto k^{-2}$, at the time of horizon crossing, the amplitudes would be $\delta_k(z_{cross}) \propto k^{-2+n/2} = k^{-\alpha}$ where $\alpha = 2 - n/2$.

The total density contrast at any location will be a superposition of modes with different wavenumbers,

$$\delta(\vec{x},z) \propto \int d^3\vec{k}\,\delta_{\vec{k}}(z)\exp(i\vec{k}.\vec{x}). \tag{41}$$

The power of such perturbations is measured by $|\delta(\vec{x},z)|^2$. So, modes in the range $(\vec{k},\vec{k}+d^3\vec{k})$ contribute an amount proportional to $d^3\vec{k}|\delta_{\vec{k}}|^2$ to $|\delta(\vec{x},z)|^2$. This contribution can be written as $d^3\vec{k}|\delta_{\vec{k}}|^2 \sim k^2dk|\delta_k|^2 \sim d(\ln k)(k^3|\delta_{\vec{k}}|^2)$, and so each logarithmic interval contributes an amount $(k^3|\delta_{\vec{k}}|^2)$. At z_{cross} this is $\propto k^{3-2\alpha} \propto M^{2\alpha/3-1}$. This shows that for $\alpha = 3/2$ (that is, for $n = 1$) all scales have equal contribution. This spectrum is scale-invariant and is predicted by some models of the early universe.

The final spectrum can be calculated numerically. For cold dark matter, for example, the final power spectrum can be written as,

$$P(k) = |\delta_k|^2 = \frac{Ak}{(1+\alpha k+\beta k^2)^2}\,, \tag{42}$$

where $\alpha = 8/(\Omega h^2)$ Mpc and $\beta = 4.7/(\Omega h^2)^2$ Mpc^2. Note that this spectrum has a break (a peak) at $k_{break} = 1/\sqrt{\beta} = 0.46\Omega_{m0}h^2$ Mpc^{-1}. The parameter combination Ωh which controls the shape of $P(k)$ is often called Γ and the best fits to observations of large scale structure suggest that $\Gamma \approx 0.3$

The story of baryons is somewhat similar. Firstly, scales smaller than the Silk scale is inhibited. Then, as photons are tightly coupled to baryons till t_{dec}, the perturbations do not grow until that time. Notice that dark matter perturbations start growing at t_{eq}, so that at decoupling the density contrast δ for dark matter is larger than that of baryons, by a factor $R_{dec}/R_{eq} \sim 21(\Omega h^2)^{-1}$. After this epoch, the baryonic perturbations are driven by the already existing dark matter perturbations. For this coupled perturbation, one can write,

$$\frac{d^2}{dt^2}\delta_B + 2\frac{\dot{R}}{R}\frac{d}{dt}\delta_B \approx 4\pi G\rho_{DM}\delta_{DM}\,. \tag{43}$$

Since $\delta_{DM} \propto R$, after t_{dec} for $\Omega = 1$, this gives a growing solution for $\delta_B = \delta_{DM}(1-\frac{A}{R})$ where A is a constant. This means that $\delta_B \to \delta_{DM}$ for $R \gg A$. So, after decoupling the baryons and dark matter perturbations grow together.

1.8 Comparison with observations

1.8.1 Correlation function and power spectrum

We have defined the density perturbations as

$$\delta(\vec{r}) = \frac{\Delta\rho}{\overline{\rho}} = \frac{(2\pi)^{3/2}}{V^{1/2}}\sum \delta_k e^{i\vec{k}.\vec{r}}\,, \tag{44}$$

where V is the volume of a box with periodic boundary conditions. The two-point correlation function of the density is given by,

$$\begin{aligned}\xi(\vec{r}) &= \langle\delta(\vec{r}')\delta(\vec{r}'+\vec{r})^*\rangle = \frac{(2\pi)^3}{V}\sum\sum\delta_k\delta^*_{k'}\langle e^{i\vec{k}.\vec{r}'}e^{-i\vec{k}'.(\vec{r}'+\vec{r})}\rangle \\ &= \frac{(2\pi)^3}{V}\sum|\delta_k|^2e^{-i\vec{k}.\vec{r}}\,. \end{aligned}\tag{45}$$

The spacing of the k's in the sum is given $\delta k_x = 2\pi/L_x$, where L_x is the length of the box in x, we have $\Delta k_x\Delta k_y\Delta k_z = (2\pi)^3/V$. The power spectrum is defined as $P(k) = \langle|\delta_k|^2\rangle$, which has the units of length^3. We then get,

$$\begin{aligned}\xi(r) &= \int P(k)e^{-i\vec{k}.\vec{r}}d^3\vec{k} = \int P(k)k^2dk\int e^{-ikr\cos\theta}d\Omega \\ &= P(k)k^2dk2\pi\int e^{-ikr\mu}d\mu = 4\pi\int k^2P(k)\frac{\sin(kr)}{kr}dk\,. \end{aligned}\tag{46}$$

Note that this equation also shows that the power spectrum is the transform of the correlation function. Since

$$\int e^{i\vec{k}.\vec{r}} d^3r = (2\pi)^3 \delta^3(\vec{r}) \,, \tag{47}$$

where the last factor is a Dirac delta function, not the density contrast, we have,

$$\int d^3k P(k) e^{i\vec{k}.\vec{r}} = \xi(r) \,. \tag{48}$$

For the above CDM power spectrum one finds the correlation function to be proportional to

$$\begin{aligned} \xi(r) &\propto \int k^2 P(k) \frac{\sin(kr)}{kr} dk \\ &\propto \Gamma^4 \int \frac{\kappa^3}{(1+\alpha'\kappa+\beta'\kappa^2)^2} \frac{\sin(\kappa\Gamma r)}{(\kappa\Gamma r)} d\kappa \\ &= \Gamma^4 F(\Gamma r) \,, \end{aligned} \tag{49}$$

with $k = \kappa\Gamma$, where

$$F(x) = \int \frac{\kappa^3}{(1+\alpha'\kappa+\beta'\kappa^2)^2} \frac{\sin(kx)}{kx} d\kappa \,. \tag{50}$$

Observations suggest that $\xi(r) = 1$ for $r = 5h^{-1}$ Mpc, and that $\xi(r) \propto r^{-1.77}$. One finds that to match this one needs $\Gamma \sim 0.3$.

We need to have more quantitative measures of the perturbation field before we can compare with observations. Let us assume that the linear density field $\delta(\vec{x},t)$ is Gaussian. We define a smoothed density field $\delta_s(\vec{x},r_s,t)$ by convolving $\delta(\vec{x},t)$ with a window function $W(x)$,

$$\delta_s(\vec{x},r_s,t) = \int \delta(\vec{x}',t) W(|\vec{x}-\vec{x}'|,r_s) d^3x' = \frac{(2\pi)^{3/2}}{V^{1/2}} \sum \tilde{W}(k) \delta_{\vec{k}} e^{i\vec{k}.\vec{x}} \,, \tag{51}$$

where the last step follows from the series expansion of δ. The Fourier transform of the window function is

$$\tilde{W}(k) = \int d^3x W(x) e^{i\vec{k}.\vec{x}} \,, \tag{52}$$

and the normalisation condition is,

$$\tilde{W}(0) = \int d^3x W(x) = 1 \,. \tag{53}$$

One example of the window function is the *top hat* function for which $W(x) = 1/V_s$ for $x \le r_s$ and 0 for $x > r_s$. This function has a sharp edge at r_s. The Fourier transform of this function is,

$$W(kr_s) = 3\left[\frac{\sin(kr_s)}{(kr_s)^3} - \frac{\cos(kr_s)}{(kr_s)^2}\right] = \frac{3}{kr_s} j_1(kr_s) \,, \tag{54}$$

where j_1 is the spherical Bessel function. The variance (mean square value) of the smoothed density contrast is,

$$\begin{aligned} \Delta^2(r_s,t) = &= \langle \delta_s(\vec{x})^2 \rangle = \frac{(2\pi)^3}{V} \sum \tilde{W}(k)^2 |\delta_{\vec{k}}|^2 \\ &= \int d^3k \tilde{W}(k)^2 P(k) \,. \end{aligned} \tag{55}$$

Note that this is dimensionless and can also be written as,

$$\Delta^2(r_s,t) = \int d^3r_1 d^3r_2 W(\vec{r}_1)W(\vec{r}_2)\xi(\vec{r}_1 - \vec{r}_2)\,. \qquad (56)$$

Consider the integral,

$$\begin{aligned} J_3(R) &\equiv \frac{1}{4\pi}\int_0^R d^3\vec{r}\xi(\vec{r}) = \int_0^R r^2 dr \xi(r) \\ &= 4\pi \int^R r^2 dr \int k^2 dk P(k) \frac{\sin(kr)}{kr}\,. \end{aligned} \qquad (57)$$

This, on integration by parts, gives,

$$J_3(r) = \frac{4\pi r^3}{3}\int_0^\infty k^2 dk P(k)\tilde{W}_s(kr)\,, \qquad (58)$$

where $\tilde{W}_s(kr)$ is the transform of a top hat window function. This means that,

$$J_3(r) = 4\pi r^3 \int k^2 dk P(k)\Big[\frac{\sin(kr)}{(kr)^3} - \frac{\cos(kr)}{(kr)^2}\Big] \approx \frac{4\pi r^3}{3}\int_0^{r^{-1}} k^2 dk P(k)\,, \qquad (59)$$

where we have ignored the contribution from wavelengths smaller than r. The expression in the square brackets has the value $1/3$ for $kr \ll 1$.

We would like to compare this with the fluctuation in mass within the volume V_s of the window function. The variance in mass distribution is essentially

$$\begin{aligned} \Delta^2(r) &= \int d^3k \tilde{W}(k)^2 P(k) = 4\pi \times 9 \int k^2 dk \Big[\frac{\sin(kr)}{(kr)^3} - \frac{\cos(kr)}{(kr)^2}\Big]^2 P(k) \\ &\approx 4\pi \int_0^{r^{-1}} k^2 dk P(k)\,, \end{aligned} \qquad (60)$$

so that we have,

$$\Delta^2(r) = \langle(\delta M/M)_r^2\rangle \approx \frac{3J_3(r)}{r^3}\,. \qquad (61)$$

Thus the integral $J_3(r)$ is a direct measure of mass fluctuation in the scale r.

The observed ξ has a cutoff at a lengthscale of $\sim 30h^{-1}$ Mpc. Evaluating the integral J_3 for such a function, one finds $J_3 \sim 100h^{-3}$ Mpc^3, which is consistent with surveys like CfA (Davis & Peebles 1983).

It has become conventional to talk about the rms amplitude of the fluctuations measured in spheres of radius $r_s = 8\,h^{-1}$ Mpc, which is denoted by σ_8. This is because early estimates of the two-point galaxy correlation function (e.g., Davis & Peebles 1983) from CfA survey suggested that $\sigma_8 \sim 1$ for optically selected galaxies. However, galaxies may not be clustered in exactly the same way as the mass. It is possible that fluctuations in the galaxy distribution is proportional to fluctuations in the mass distribution,

$$(\delta\rho/\rho)_g = b(\delta\rho/\rho)_\rho\,, \qquad (62)$$

where b is a constant, called the 'biasing factor'.

For example, from COBE data one estimates σ_8 for the standard CDM to be 1.2 (Bunn & White 1996). If the biasing is linear, then $\xi_\rho = b^2\xi_g$ and $P_\rho(k) = b^2 P_g(k)$, which means that $\sigma_\rho(r_s) \approx 1/b$ for $r_s = 8\,h^{-1}$ Mpc. Note that the observed correlation function mentioned earlier is the galaxy-galaxy correlation function.

This 'biased' galaxy formation scenario essentially means that light may not always trace mass, since there may be galaxies that are either too faint to be seen or never lit up. The idea of biased galaxy formation relies upon some of the fundamental statistical properties of the density field itself. Consider the density field smoothed with a top hat window function of radius r_s which is appropriate for a galaxy ($r_s \sim 1$ Mpc), with rms variation given by $\Delta(r_s) = (\delta M/M)_g$. The probability of having a density contrast δ at a given point in space is proportional to $\exp(-\delta^2/2\Delta^2)$ if the field is Gaussian. In some regions of space the value of δ will exceed Δ, which means that some galactic sized perturbations are particularly overdense. These perturbations will obviously collapse and form galaxies before the more common-sized perturbations (say, those with $\delta \sim \Delta$). Crudely speaking, the 3σ high-density peaks ($\delta = 3\Delta$) grow into galaxies first, then the 2σ peaks and so on. Different sized perturbations can be labeled by $\nu = \delta/\Delta$. Now, it turns out that the correlation function is proportional to ν^2, that is, galaxies that form form higher density peaks are more strongly correlated than galaxies from lower density peaks, or, in general, from the underlying mass density field itself.

To understand this, consider a sinusoidal wave with wavelength $\lambda \sim$ Mpc, whose amplitude varies in space and is distributed about the mean value $\overline{\delta}$ with a Gaussian distribution. Regions with amplitude values which are much greater than $\overline{\delta}$ are very rare. Now imagine superimposing this wave on a longer wavelength sine wave. Statistically it will now be easier to exceed a given threshold in amplitude, say $\nu_{rh} \gg 1$ by riding on one of the crests of the longer wavelength perturbation, and so peaks that exceed the threshold ν_{th} will be preferentially found on the crests of the underlying longer wavelength wave. So the high density peaks will be strongly clustered.

One down-to-earth analogy is that of clustering of mountain peaks. Mountains of height more than 25000 ft are strongly clustered in the Himalayas.

So, if the galaxy formation process requires that there is a threshold ν for forming bright galaxies, then the correlation function ξ_g found from bright galaxies will be ν_{th}^2 times the correlation function of the underlying mass density. This is basically the biased galaxy formation scenario. One therefore has a parameter, *b* the biasing factor, to fit the observed correlation function or the power spectrum to the theoretical values or to the COBE normalization (which depends on the underlying mass fluctuation at much larger scales than where galaxy formation processes matter).

The power spectrum of galaxy distribution has been determined from different redshift surveys. In general one finds that if one normalizes the CDM power spectrum at small scales, it cannot fit the observed spectrum at large scales. If one normalizes it with COBE (see below) then one overproduces small scale structure, which means that if CDM is correct that there must be some hidden population of nearby galaxies. One also finds that an open universe or a universe with non-zero cosmological constant fits the data best. One intriguing feature of the recent Las Campanas survey is the spike in the power spectrum around $100h^{-1}$ Mpc scale. It is still an open question whether or not it really exists and if so what causes it.

1.8.2 CMBR anisotropy

Sachs and Wolfe (1967, ApJ, 147,73) showed that a gravitational potential perturbation produces an anisotropy of the CMBR with an amplitude,

$$\frac{\Delta T}{T} = \frac{1}{3}\frac{\Delta\phi}{c^2}, \tag{63}$$

where $\Delta\phi$ is perturbation in potential at the intersection of the line of sight and the last scattering surface. The anisotropy is usually expressed in terms of spherical harmonics,

$$\frac{\Delta T(\hat{n})}{T} = \sum_l \sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{n})\,. \tag{64}$$

It is similar to Fourier decompositions of functions in flat space into since and cosines– this is on a sphere. If there is only a dipole then the decomposition will be a delta function at $l = 1$.

Since the universe is approximately isotropic, the probability densities for all m's for a given l are identical and also the expected value of $\Delta T(\hat{n})$ is zero, and thus the expected value of a_{lm}'s is zero. But the variance of a_{lm} is not zero and is defined as ,

$$C_l = \langle |a_{lm}|^2 \rangle\,. \tag{65}$$

Here $l \sim 180°/\theta$. It turns out that for a scale-invariant matter power spectrum ($n = 1$), the radiation power spectrum is (Bond & Efstathiou 1987,

$$C_l = \frac{4\pi Q^2}{5T_0^2} \frac{6}{l(l+1)}\,, \tag{66}$$

where Q^2 is the variance of the $l = 2$ component of the sky. Since the number of spherical harmonics contribuing to the anisotropy power at an angular scale θ is $l(2l+1)$, this means that for a scale-invariant power spectrum, the anisotropy power at each scale is equal. The COBE DMR experiment has found $\sqrt{\langle Q^2 \rangle} = 18\mu$K, and also that C_l's from $l = 2$ to $l = 20$ are consistent with the above prediction. For scale invariant spectrum, $|\delta_k|^2 \propto k$, so that the density fluctuations on a comoving scale L scale as $\delta\rho_L \propto L^{-2}$, and the associated potential fluctuation on a physical scale $\lambda = R(t)L$ are $(\Delta\phi)_\lambda \sim \frac{G\delta M}{\lambda} \sim R^2(t)G\rho L^2$ which are independent of L. So, the temperature fluctuation $\Delta T/T$ is also independent of θ for a scale-invariant spectrum.

The angular correlation function of the anisotropy is given by

$$C(\theta) = \frac{\langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle}{T_0^2} = \frac{1}{4\pi} \sum_l (2l+1) C_l P_l(\cos\theta)\,, \tag{67}$$

where P_l is a Legendre polynomial.

Let us try to find out the connection between the matter power spectrum and C_l so that we can normalize the matter power spectrum. We have from the last line of equation (6) that,

$$\phi_k = -4\pi G \bar{\rho} R^2 \delta_k k^{-2}\,. \tag{68}$$

The correlation function for the potential fluctuations can be written as,

$$C_\phi(r) = 4\pi \int k^2 \phi_k^2 \frac{\sin(kr)}{kr} dk = 64\pi^3 G^2 \bar{\rho}^2 R^4 \int P(k) k^{-2} \frac{\sin(kr)}{kr} dk\,. \tag{69}$$

For super-horizon scale perturbations, $\bar{\rho}R^2\delta$ is constant and we can write it as $\rho_{crit}(t_0)\delta(t_0)$ for a $\Omega = 1$ universe. Using $\Delta T/T = \phi/3c^2$ gives the $\Delta T/T$ correlation function at t_0 as,

$$C(\theta) = \frac{C_\phi(r)}{9c^4} = \pi \Big(\frac{H_0}{c}\Big)^4 \int P(k) k^{-2} \frac{\sin(kr)}{kr} dk\,. \tag{70}$$

Here $r = 2R_{LS}\sin(\theta/2)$ and $R_{LS} = (2c/H_0)(1 - 1/\sqrt{1+z_{LS}})$ is the comoving radius of the surface of last scattering.

The expansion of a plane wave into spherical harmonics,

$$
\begin{aligned}
e^{i\vec{k}.\vec{r}} &= 4\pi\sum_{l=0}^{\infty} i^l j_l(kr)\sum_{m=-l}^{l} Y^*_{lm}(\theta,\phi)Y_{lm}(\theta,\phi) \\
&= \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\gamma)\,, \qquad (71)
\end{aligned}
$$

Here $j_l(x)$ is a spherical Bessel function, (θ,ϕ) is the direction of $\vec{r}$, (θ',ϕ') is the direction of $\vec{k}$ and γ is the angle between the two directions. The density correlation between two points $\vec{r}_1$ and $\vec{r}_2$ which are both situated on the last scattering surface is ($|\vec{r}| = R_{LS}$, where R_{LS} is the comoving radius of the last scattering surface),

$$
\begin{aligned}
\xi &= \langle\delta(\vec{r}_1)\delta(\vec{r}_2)^|ast\rangle = \frac{(2\pi)^3}{V}\sum_k\sum_{k'}\delta_k\delta^*_{k'}\langle e^{i\vec{k}.\vec{r}_1}.e^{-i\vec{k}'.\vec{r}_2}\rangle \\
&= \frac{(2\pi)^3}{V}\sum_k|\delta_k|^2 e^{i\vec{k}.\vec{r}_1}.e^{-i\vec{k}'.\vec{r}_2} \\
&= \frac{(2\pi)^3}{V}\sum_k|\delta_k|^2 4\pi\sum_{l=0}^{\infty} i^l j_l(kR_{LS})\sum_{m=-l}^{l} Y^*_{lm}(\theta_1,\phi_1)Y_{lm}(\theta',\phi') \\
&\times\ 4\pi\sum_{l'=0}^{\infty} i^{l'} j_{l'}(kR_{LS})\sum_{m'=-l'}^{l'} Y^*_{l'm'}(\theta_2,\phi_2)Y_{lm}(\theta',\phi') \qquad (72)
\end{aligned}
$$

We will convert the sum over k into an integral, which will include an integral over (θ',ϕ') which will force $l = l'$ and $m = m'$ since,

$$
\int Y_{lm}(\theta',\phi')Y^*_{l'm'}(\theta',\phi')d\Omega' = \delta_{ll'}\delta_{mm'}\,, \qquad (73)
$$

and so we will get,

$$
\xi = \int k^2P(k)\big[4\pi\sum_{l=0}^{\infty} j_l^2(kR_{LS})4\pi\sum_{m=-l}^{l} Y^*_{lm}(\theta_1,\phi_1)Y^*_{lm}(\theta_2,\phi_2)\big]dk\,. \qquad (74)
$$

But the sum of spherical harmonics is

$$
4\pi\sum_{m=-l}^{l} Y^*_{lm}(\theta_1,\phi_1)Y^*_{lm}(\theta_2,\phi_2) = (2l+1)P_l(\cos\gamma)\,, \qquad (75)
$$

and so,

$$
\xi = 4\pi\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma\int k^2P(k)j_l^2(kR_{LS})dk\,. \qquad (76)
$$

We get the correlation function of temperature by multiplying as before by $\left(\frac{4\pi G\bar{\rho}R^2}{3k^2c^2}\right)^2 = (1/4)(H_0/c)^4k^{-4}$ and get,

$$
C(\gamma) = \pi\Big(\frac{H_0}{c}\Big)^4\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\int k^{-2}P(k)j_l^2(kR_{LS})dk\,, \qquad (77)
$$

which readily gives the angular power spectrum (using equation (67)),

$$
C_l = 4\pi^2\Big(\frac{H_0}{c}\Big)^4\int k^{-2}P(k)j_l^2(kR_{LS})dk\,. \qquad (78)
$$

For a power spectrum of type $P(k) = Ak$ one gets for the quadrupole,

$$C_2 = 4\pi^2 \Big(\frac{H_0}{c}\Big)^4 A \int P(k)\, j_2^2(kR_{LS}) \frac{dk}{k} = 4\pi^2 \Big(\frac{H_0}{c}\Big)^4 A \int j_2^2(x) \frac{dx}{x} = \frac{\pi^3}{3} \Big(\frac{H_0}{c}\Big)^4 A. \quad (79)$$

So from the measurement of C_2 (from Q) one can normalize the power spectrum.

Roughly the anisotropy of the CMBR is of order $\Delta T/T \sim \Delta\phi/c^2 \sim v_c^2/c^2$ where v is the velocity dispersion in the perturbation. For a typical cluster of comoving size ~ 5 Mpc, $v_c \sim 1000$ km/s. Also, a comoving scale L at $z \gg 1$ subtends an angle $\theta \sim (L/Mpc)(\Omega_0 h)$ arcminutes. So that a typical cluster would produce a temperature fluctuation of order 10^{-5} at an angular scale of $5'$. COBE has determined the fluctuations at a much larger scale, but the values are consistent with the predictions from the idea that structures form from gravitational instability. There are many other experiments going on at present and planned for future to probe the fluctuations at various scales—a nice place to look at for an introduction to these experiments is http://www.hep.upenn.edu/ max/index.html, and for theory, have a look at http://www.sns.ias.edu/ whu/physics/physics.html.

1.8.3 Variations on the CDM theme

Since CDM cannot exactly fit the observations , although it remains an attractive model since it provides more or less the correct shape of the spectrum, a few variations of the CDM have turned up recently. One talks of Mixed Dark Matter (MDM) in which one mixes abit of hot dark matter ($\sim$ 10% to 30%), to get the power at large scales while keeping enough CDM for the small structures. One also talks of a tilted CDM, for which n deviates slightly from unity, essentially to decrease the power at small scale. And then one talks of a non-zero Λ universe. If most of the contribution to $\Omega = 1$ comes from the Λ term, then the lower Ω leads to lower matter density, which means that it would take longer for the universe to reach the matter-radiation equality, which gives the universe more time to wash out small scale fluctuations. (Note that this is also achieved by lowering the value of H_0 and a value of $h \sim 0.3$ can fit the data very well indeed.) For such a Λ-universe, the age is larger for a given H_0, which also helps in solving some problems like ages of globular clusters. This is why a Λ- universe has become very attractive these days (notwithstanding the recent results from the SN 1a searches at high redshift).

1.8.4 Clusters of galaxies

Observations of galaxy clusters provide a number of clues for the cosmological parameters in the context of structure formation. For example, the fact that substructures are seen in clusters argue against low value of Ω since it means that formation of clusters is still ongoing or at least have terminated recently. The formation epoch of clusters is also a strong constraint— one does not expect much clustering at high redshift in CDM type models. So the recent findings that clustering can be high at $z \sim 2$–3 can go against CDM models, although one should remember that in a biased galaxy formation scenario, the galaxies which form first are necessarily from very high density peaks and which are strongly correlated.

Identifying the cluster formation epoch is not a straightforward task. One finds substructures in nearby clusters, which show that they are still accreting material. In that case the formation of cluster is still going on. But then there are evidences that substructures existed even as far back as $z \sim 0.9$ (Postman et al.1996, AJ, 111, 615) which look much like the core of Coma cluster now.

Observations of the hot intra-cluster medium (ICM) which emits X-rays (with temperatures of 0.5–5 keV provides some constraints on the formation epoch of clusters. The origin of the ICM is still unclear though. It is probably a mixture of leftover gas that did not get incorporated into any galaxies, or gas that was driven out of galaxies by supernovae or tidally liberated gas from galaxies. The ICM is also highly enriched with metallicities of order 0.5 $Z_\odot$ which shows the dominance of material processed within galaxies and later driven out.

If the ICM is heated to the virial temperature by the cluster potential then observations of the evolution of the X-ray luminosity function of clusters as a function of redshift can reveal the epoch of cluster formation. This is being done now with the help of data from ROSAT and ASCA. One can now find X-ray emission from clusters even at $z \sim 1$ (Hattori et al.1997). The Chandra X-ray Observatory (to be launched in 1999 Spring) will have enough sensitivity to go further in redshift.

There are other aspects of clusters which serve as constraints on structure formation models. Zabludoff & Geller (1994) and Crone & Geller (1995) claim that one can constrain the structure formtaion model with the observation of abundance of clusters with a threshold value of velocity disperson, and models that fit the data best have either $\Omega \sim 0.2$ or is a biased $\Omega = 1$ model.

1.8.5 High redshift galaxies

Recent observations with both ground based telescopes and HST have discovered galaxies at high redshifts ($z \sim 5$). With the help of the Lyman break in the galactic spectrum which gets shifted to different bands depending on the redshift, one can identify high redshift galaxies by observing a field in different bands and comparing the pictures in different colours. Steidel et al.(1996, ApJL, 462, L17) first discovered with ground based observations a bunch of star-forming galaxies at $z \sim 3$, and several discoveries soon followed. Then the observations of the Hubble Deep Field (North, and now also South) yielded several galaxies at high redshift. Lanzetta et al.(1996, Nature, 381, 759) claimed that the reddest objects in HDF(North) have redshifts $z > 6$, although the confirmation would need spectroscopic data.

A few things are clear: (1) The top down scenario of a neutrino dominated model is ruled out as one does not expect such high abundance of high redshift galaxies in that model. (2) The Morphology of high z galaxies gives the impression that galaxy formation occurs via merging of small subunits, which is a confirmation of the idea of hierarchical structure formation models. (3) Mo and Fukugita (1996, ApJL, 467, L9) have argued that to reproduce the abundance of Lyman-break galaxies one needs a Λ-dominated universe.

To understand the last point, one needs a quantitative measure of the abundance of objects in a given structure formation model, and this is provided by the Press-Schechter mass function. Press & Schechter (1974) derived an expression for the mass function assuming that non-linear clumps could be identified as overdensities in the linear density field. They argued that if the overdensity at any point exceeded a critical threshold δ_c when smoothed with a top-hat filter of radius r_s (at some epoch z_i), then a mass element would would be incorporated in a non-linear object of mass M or greater by some epoch z, where $M = (4\pi/3)\rho_{average} r_s^3$. To be precise, δ_c is a function of the epoch z at which one wants to determine the mass function and z_i. If the density field is assumed to be Gaussian, then the probability that the density field will have a value

δ at any chosen point is given by,

$$P(\delta,t) = \Big(\frac{1}{2\pi\Delta^2(r_s,t)}\Big)^{1/2} \exp\Big(-\frac{\delta^2}{2\Delta^2(r_s,t)}\Big)\,. \tag{80}$$

So from the above arguments, one can write the fraction of bound objects with mass greater than M as,

$$\begin{aligned} F(>M) &= \int_{\delta_c(t,t_i)}^{\infty} P(\delta,r_s,t_i)\,d\delta = \frac{1}{\sqrt{2\pi}}\frac{1}{\Delta(r_s,t_i)}\int_{\delta_c}^{\infty}\exp\Big(-\frac{\delta^2}{2\Delta^2(r_s,t_i)}\Big) \\ &= \frac{1}{2}\mathrm{erfc}\Big(-\frac{\delta_c(t,t_i)}{\sqrt{2}\Delta(r_s,t_i)}\Big)\,, \end{aligned} \tag{81}$$

where $\mathrm{erfc}(x)$ is the complementary error function. The fraction of the mas density in non-linear objects of mass M to $M+dM$ is given by the derivative of $F(>M)$, that is $f(M) = (\partial F/\partial M)$. The comoving number density $N(M,t)$ can be found by dividing this expression by $M/\overline{\rho}$ and one gets,

$$N(M,t)dM = -\Big(\frac{\overline{\rho}}{M}\Big)\Big(\frac{1}{2\pi}\Big)^{1/2}\Big(\frac{\delta_c}{\Delta}\Big)\Big(\frac{1}{\Delta}\frac{d\Delta}{dM}\Big)\exp\Big(-\frac{\delta_c^2}{2\Delta^2}\Big)dM\,. \tag{82}$$

One can easily determine this given the structure formation model, that is the power spectrum and the window function.

There is a problem though with this mass function. Although the integral of $f(M)$ over M should give unity, the expression above for $f(M)$ yields $1/2$ after integrating, that is, $\int_0^\infty f(M)dM = 1/2$. This is because we have not considered the underdense regions correctly. Consider a region with $\delta < \delta_c$. There is a non-zero probability that such regions would have $\delta > \delta_c$ when the density field is smoothed with a window with $r_{s1} > r_s$. These points should also correspond to regions with mass greater than M. But the above mass function ignores these regions and so underestimates $F(M)$. Press & Schechter (1974) solved this problem by multiplying the mass function by a factor of 2. So, the comoving number density of objects in the Press-Schechter model is given by,

$$N(M,t)dM = -\Big(\frac{\overline{\rho}}{M}\Big)\Big(\frac{2}{\pi}\Big)^{1/2}\Big(\frac{\delta_c}{\Delta}\Big)\Big(\frac{1}{\Delta}\frac{d\Delta}{dM}\Big)\exp\Big(-\frac{\delta_c^2}{2\Delta^2}\Big)dM\,. \tag{83}$$

This ‘fudge factor’ of 2 has, however, been justified in a more rigorous analysis by Bond et al.(1991).

The critical contrast can be estimated in the spherical collapse model. Consider a spherical shell containing mass M collapsing under gravity,

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}\,, \tag{84}$$

and the solution is,

$$r = A(1-\cos\eta)\;t = B(\eta - \sin\eta)\,A^3 = GMB^2\,. \tag{85}$$

This shell collapses to zero radius at $\eta = 2\pi$ when the time is $t_c = 2\pi B$. The mean physical density within the shell is $\overline{\rho} = 3M/4\pi r^3$, the mean density in the background universe (for $\Omega = 1$) is $\rho_b = 1/(6\pi G t^2)$, and the ratio gives the density contrast. At $\eta \ll 1$, for small values of contrasts, the series expansion gives,

$$\delta = \frac{\overline{\rho}}{\rho_b} - 1 = \frac{3\eta^2}{20} = \frac{3}{20}\Big(\frac{12\pi t}{t_c}\Big)^{2/3}\,. \tag{86}$$

This shows that the density contrast extrapolated to $t = t_c$ in linear perturbation theory is $\delta_c \to 1.69$.

Note that for a fluctuation power spectrum with a power law, $|\delta_k|^2 \propto k^n$, one has $\Delta^2(r_s) \propto r_s^{-(3+n)} \propto M^{-(3+n)/3}$ and, so

$$\frac{M^2 f(M)}{\overline{\rho}} = \sqrt{\frac{2}{\pi}}\frac{\delta_c}{\Delta}|\frac{d\ln\Delta}{d\ln M}|\exp(-\frac{1}{2}\delta_c^2/\Delta^2) = \frac{n+3}{6}\sqrt{\frac{2}{\pi}}\nu e^{-\nu^2/2}, \tag{87}$$

where $\nu = \delta_c\Delta$ is the threshold in the units of the rms density fluctuation. So the mass function has the same shape (with a hump at $\nu \sim 1$) and the value of ν changes with time, with the hump shifting towards larger mass scales with time. (Since $\nu \propto M^{(n+3)/6}$, one writes $\nu = (M/M_c)^{(n+3)/6}$ and identifies M_c as the characteristic mass scale where the mass function steepens.

This characteristic mass scale, however, needs to be altered if one discusses galaxy formation from dissipative baryonic matter. The PS formalism is correct only for collisionless dark matter and does not take the constraint that baryonic matter needs to cool in order to form galaxies (Rees & Ostriker 1977, MNRAS 179, 541). Peacock & Heavens (1990, MNRAS, 243, 133) has taken this into account and showed that this changes the mass function slope to $M^2 f(M) \propto M^{(n+3)/6+2/3}$.

The Press-Schechter mass function has always been treated with some scepticism, as it is yet to be tested in great detail in all regimes of mass. However, it seems to perform quite well when compared with N-body simulations, at least for masses of order $M > 10^{11} M_\odot$.

At any rate, this mass function has been used to compare the structure formation models with respect to abundance of galaxies at high redshift. One must remember that the process of galaxy formation is a complicated one, involving heating and cooling of gas due to various processes, and non-linear phenomena like shocks, and the lack of understanding of these processes makes such constraints weak.

1.8.6 Peculiar velocities

To discuss the velocity field in linear perturbation theory, let us assume that the growing mode dominates and write $\delta = A(\vec{x})D(t)$, where $D(t)$ is the growing mode solution of the fluctuation. the the mass conservation equation is (equation (4)),

$$\nabla.\vec{v} = -R\frac{\partial\delta}{\partial t} = -R\delta\frac{\dot{D}}{D}. \tag{88}$$

Here ∇ is in the comoving frame and $\vec{v} = R\vec{u}$. Let us write the velocity field as the sum of a part with no divergence and an irrotational part. The first part then plays no part in the evolution of the density contrast in the conservation equation, and this component decays as R^{-1} from the Euler's equation. The above equation is the Poisson equation for the irrotational part and the familiar solution from electrostatistics is,

$$\vec{v}(\vec{x}) = \frac{R}{4\pi}\frac{\dot{D}}{D}\int\frac{\vec{y}-\vec{x}}{|\vec{y}-\vec{x}|^3}\delta(\vec{y})d^3y = \frac{RfH}{4\pi}\int\frac{\vec{y}-\vec{x}}{|\vec{y}-\vec{x}|^3}\delta(\vec{y})d^3y, \tag{89}$$

where the dimensionless factor,

$$f = \frac{R}{\dot{R}}\frac{\dot{D}}{D} = \frac{1}{H}\frac{\dot{D}}{D}. \tag{90}$$

Comparing the Poisson equation (the last of equation (4)) with equation (88) and remembering that the perculiar graviational acceleration $\vec{g} = -\nabla\phi/R$, one can write the peculiar velocity as,

$$\vec{v} = \frac{fH}{4\pi G\bar{\rho}}\vec{g} = \frac{2}{3}\frac{f}{\Omega H}\vec{g}. \tag{91}$$

In an $\Omega = 1$ universe, $f = 1$ and $H = 2/(3t)$ and the peculiar velocity field has the simple form, $\vec{v} = \vec{g}t$. For a spherical mass fluctuation, one has from (equation (89))

$$v(x) = -R\frac{fH}{x^2}\int_0^x y^2 dy\delta(y) = -\frac{1}{3}fHRx\bar{\bar{\delta}}, \tag{92}$$

where $\bar{\bar{\delta}}(x)$ is the mass density contrast averaged within the radius x. The factor f depends primarily on Ω and a useful approximation is that $f \sim \Omega^{0.6}$.

Since the peculiar velocity field depends on the density contrast and Ω, one can try to match the observed velocity field with that derived from theory and try to constrain the density contrast and Ω. Analysis with the local velocity field seems to indicate that $\Omega \sim 0.3$ (Dekel 1994 ARAA).

1.8.7 The intergalactic medium

Another interesting set of constraints comes from the study of absorption lines in the spectra of QSOs. Firstly from the lack of any absorption trough shortward of the Lyman emission line in the QSO spectra, one can put some limits on the abundance of neutral hydrogen (HI) in the intergalactic medium. For an $\Omega = 1$ universe with a homogeneously distributed intergalactic medium (IGM) with neutral fraction f, the opacity (the so-called Gunn-Peterson opacity) is,

$$\tau_{GP} = 4.6 \times 10^5 \Omega_{IGM} h(1-f)(1+z)^{2.5}. \tag{93}$$

The fact that $\tau < 0.1$ at $z \sim 4$ means that, $\Omega_{HI} < 2 \times 10^{-8} h^{-1}$. This shows that the universe was reionized at some epoch $z > 4$.

However, with the recent results from numerical simulations, it is clear that in addition to reionization the fragmentation of the IGM also contributes to the lack of the Gunn-Peterson absorption. Although there is no absorption trough, there are discrete absorption lines, with a large range in the HI column densities. These are now thought to arise (lines with HI column densities $N_{HI} < 10^{14}$ cm^{-2}) from small scale fluctuations in matter. Therefore the study of these lines now provide important clues to the process of structure formation. Recent works have focussed on the possibility of recovering the power spectrum from observations. What they do is this: (1) find a emperature evolution law from simple considerations of photoinoization heating and cooling, (2) find a relationship between the HI content from photoionization equilibrium ($n_{HI} \propto \rho^2 \Gamma_{HI} T^{-0.7}$ where Γ_{HI} is the ionization rate) and the overdensity, and (3) then a relation between the opacity (lack of flux) and the overdensity. This comes out to be like $\tau \propto (\rho/\bar{\rho})^{1.6}$ and the proportionality constant depends on the reionization history, photoionization background and the cosmological parameters. Now, this monotonic relation can be used to invert the observed spectra into giving the underlying density (1D) field, which then can be used to get the power spectrum. Recently Croft et al.have used 19 QSO spectra to recover the power spectrum and compare with theoretical models. IGM constraints give the power spectrum at comoving scales of $k \sim 1$–10 h Mpc^{-1} where it cannot be probed by galaxy surveys. The power spectrum is consistent with CDM type models and favours either an open or a Λ dominated universe.

Absorption lines with large column densities ($N_{HI} > 10^{20}$ cm^{-2}) are thought to arise from galaxies (or protogalaxies) at high redshifts. They are called the Damped Lyman-α systems. These lines are also associated with a number of metal lines, and the metal abundance of such absorption systems as a function of redshift provides important constraints to the models of structure formation and galaxy evolution.