Gravity

In the Universe at large, distances are immense. The forces that are most important in astronomical scales are long-range. This could include electric and gravitational forces, but electric forces can be shielded by bringing charges of opposite sign together – by and large most astronomical objects are charge neutral. Gravitational forces, however, cannot be shielded since masses of opposite sign do not exist. Gravity, therefore, is by far the most important force in Astrophysics.

The correct description of gravity at a classical level, as we know today, is given by the General Theory of Relativity. In our everyday experience, we are familiar with the Newtonian description of gravity, which works well in the weak field limit. In astronomy, we come across situations where a very large amount of mass could be concentrated into a small volume, where Newtonian description breaks down, or even at the weak field limit, very large distances and long lengths of time make small departures from Newtonian behaviour of gravity visible to us.

Let us, then, begin with a brief description of some results from General Theory of Relativity which we will need to use time and again through this course.

GTR describes gravity as a curvature of the four-dimensional spacetime manifold. In the absence of gravity, spacetime is "flat", and the squared "distance" between two infinitesimally separated events is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

where c is the speed of light in vacuum, which is the same for all observers.

Note that along the path of a light ray $ds^2 = 0$, which defines a "light cone". "Distance" between events is measured with respect to this cone.

In the presence of gravitating masses, the geometry of spacetime departs from flatness. In general the squared infinitesimal distance is written as

$$ds^2 = g_{ij}dx^idx^j;\ i, j = 0, 1, 2, 3;\ x^0 \equiv ct$$

of which flat space is a special case with

$$g_{00} = +1, g_{0\alpha} = 0, g_{\alpha\beta} = -\delta_{\alpha\beta};\ \alpha, \beta = 1, 2, 3$$

The curvature of the spacetime can be quantitatively expressed through a "curvature tensor" constructed out of the coefficients g_{ij}, and can, in turn, be determined from the energy-momentum tensor of the matter present, using a relation called the Einstein equation. The components g_{ij} of the so-called metric tensor are thus determined by the gravity of the matter present.

Particles move along geodesics in the spacetime manifold. Once g_{ij} are known, the geodesics can be described by the equation:

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

where

$$\Gamma^l_{mn} = \frac{1}{2}g^{lk}\left[\frac{\partial g_{km}}{\partial x^n} + \frac{\partial g_{kn}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^k}\right]$$

and g^{ij} is the inverse of g_{ij}.

Light propagates along special geodesics along which $ds^2 = 0$. These are called "null geodesics".

If the field is weak, GR yields Newtonian gravity. In cartesian coordinates, the metric tensor g_{ij} in this limit is diagonal, with components

$$g_{00} = \left(1 - \frac{2\Phi}{c^2}\right);\ g_{11} = g_{22} = g_{33} = -\left(1 + \frac{2\Phi}{c^2}\right)$$

where Φ is the Newtonian gravitational potential. This shows the simplest relation between gravity and the metric coefficients.

The first Post-Newtonian correction comes in the order Φ^2/c^4.

Even if the spacetime is curved, an observer located anywhere on it can always define a local Lorentz frame in which

$$ds^2 = c^2d\tau^2 - dx^2 - dy^2 - dz^2 (= g_{ij}dx^i dx^j)$$

So a "local" time interval $d\tau = \sqrt{g_{00}}dt$ and a "local" spatial interval $dl = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{c^2d\tau^2 - ds^2}$. In the above example

$$d\tau = \left(1 - \frac{2\Phi}{c^2}\right)^{1/2} dt$$

$$dx = \left(1 + \frac{2\Phi}{c^2}\right)^{1/2} dx^1$$

$$dy = \left(1 + \frac{2\Phi}{c^2}\right)^{1/2} dx^2$$

$$dz = \left(1 + \frac{2\Phi}{c^2}\right)^{1/2} dx^3$$

where t, x^1, x^2, x^3 are coordinates defined by an observer at infinity (far away from the gravitating masses).

Clearly, the clock of the local observer runs faster than that of the observer at infinity ($d\tau < dt$). So the observed frequency of a photon, for example, will reduce as it propagates from a "local" point to infinity. The frequencies will be related by

$$\nu_\infty = \left(1 - \frac{2\Phi}{c^2}\right)^{1/2} \nu$$

This is called Gravitational Redshift.

If the spacetime is spherically symmetric (e.g. when gravity is caused by a single point mass), the metric can be written as

$$ds^2 = g_{00}(c^2 dt^2) - \left[\bar{g}_{rr} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

In General Relativity, the gravity of a point mass M located at $r = 0$ is described by

$$g_{00} = \left(1 - \frac{2GM}{c^2 r}\right)$$

$$\bar{g}_{rr} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

where G is the Newtonian Gravitational Constant. This is called the Schwarzscild metric.

The coordinates in "Local Lorentz Frame" in the Schwarzschild metric have then the following scaling with respect to those defined by a distant observer:

$$d\tau = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt$$

$$dx = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr,$$

the definitions of θ and ϕ being the same in both systems.

The acceleration of a test particle in the Schwarzschild spacetime works out to be

$$F = \frac{GM/r^2}{(1 - 2GM/c^2 r)^{1/2}}$$

From the above, we notice that $d\tau \to dt$, $dx \to dr$ and $F \to GM/r^2$ as $r \to \infty$. But as $r \to 2GM/c^2 \equiv r_g$, $d\tau \to 0$ and $F \to \infty$.

If the gravitating mass is indeed concentrated within the gravitational radius r_g, then from the region $r \leq r_g$ nothing, including light, can escape to infinity. Such an object is called a "Black Hole" and r_g is called the "Event Horizon". A black hole can have at most three attributes: its mass, angular momentum and charge. The Schwarzschild solution describes a non-rotating, uncharged black hole. If the black hole is rotating, it is described by a solution known as the Kerr solution and in case it has an overall charge (which is unlikely in the real world, since opposite charges can fall into the black hole and neutralise it), it is described by the Reissner-Nordström solution. We will not go into the details of Kerr Black Holes or the charged variety in these lectures.

Particle orbits

Let us now examine particle trajectories in the Schwarzschild spacetime. Like in the corresponding Newtonian case, conservation of angular momentum ensures that the orbits are always planar, containing the initial velocity vector and the centre of force. For a given particle, if we define this plane to be $\theta = \pi/2$, the geodesic equations can be written in the following simplified form, in terms of normalised coordinates:

$$\left(\frac{d\bar{x}}{d\tau}\right)^2 = \frac{1}{\bar{E}^2}\left[\bar{E}^2 - 1 + \frac{1}{\bar{r}} - \frac{\bar{a}^2}{\bar{r}^2} + \frac{\bar{a}^2}{\bar{r}^3}\right]$$

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{\bar{a}^2}{\bar{E}^2\bar{r}^4}\left(1 - \frac{1}{\bar{r}}\right)$$

where $\bar{E} = E/mc^2$, E being the total energy of the particle (of rest mass m) including the rest energy; $\bar{r} = r/r_g$; $d\bar{x} = (1 - 1/\bar{r})^{-1/2} d\bar{r}$ and $\bar{a} = a/mcr_g$, where a is the angular momentum of the particle. τ is the time measured by a local observer in units of r_g/c. These two equations determine particle orbits in the Schwarzschild spacetime.

Setting $d\bar{x}/d\tau = 0$ one could obtain an effective radial potential for particle motion:

$$\bar{E}(\bar{r}) = \left[\left(1 - \frac{1}{\bar{r}}\right)\left(1 + \frac{\bar{a}^2}{\bar{r}^2}\right)\right]^{1/2}$$

One could compare this with the effective radial potential expected in Newtonian description of gravity:

$$\bar{E}(\bar{r}) = 1 + \frac{1}{2}\left(\frac{\bar{a}^2}{\bar{r}^2} - \frac{1}{\bar{r}}\right)$$

At large r these two yield the same result, but as $r \to r_g$ ($\bar{r} \to 1$), they differ significantly. Figure 1 compares these two cases.

A given energy $\bar{E}$ of a particle is represented by a horizontal line in the figure. The places where this horizontal line meets the potential curve mark the turning points in the orbit. If $\bar{E} < 1$, there are two turning points which correspond to the two extremities of an elliptical orbit. At the minimum of the potential, these two extremities merge and one gets a circular orbit. Interestingly, if $\bar{a}^2 < 3$, no such minimum exists in the Schwarzschild case, indicating that stable circular motion is then not possible. This also yields the size of the smallest stable circular orbit: $\bar{r} = 3$. In the Newtonian case the rising branch of the potential is always present

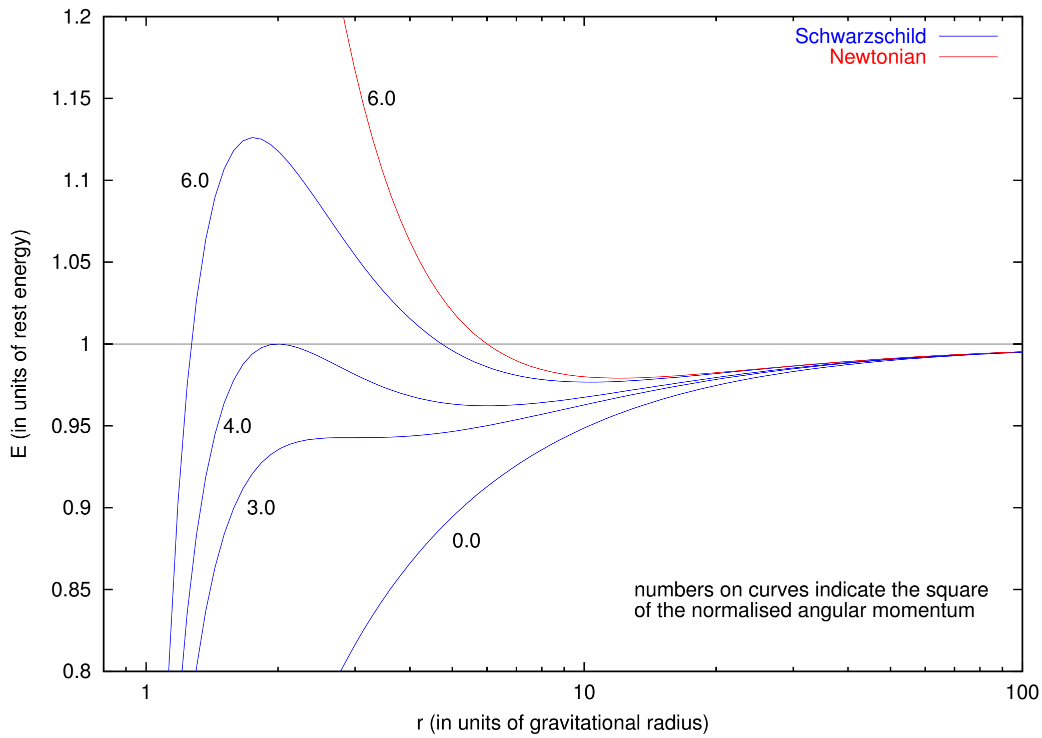

Figure 1: Radial potential curves $\bar{E}$ vs $\bar{r}$ for particles with non-zero rest mass in Schwarzschild (blue) and Newtonian (red) description of gravity. The Schwarzschild cases are shown for several values of $\bar{a}^2$. $\bar{E} > 1$ indicate positive-energy, unbound orbits, whereas $\bar{E} < 1$ correspond to bound orbits. However, in the Schwarzschild case, if the energy exceeds the upper turning point the particle is gravitationally captured.

for any $\bar{a} > 0$, and the smallest stable circular orbit has zero radius. For $\bar{a}^2 > 3$, the Schwarzschild potential curves have also a maximum. If the energy of the particle exceeds this maximum, the particle is gravitationally captured. If it is lower than, but very close to the maximum then $d\bar{x}/d\tau$ approaches zero very slowly, while $d\phi/d\tau$ remains finite, causing the particle to undergo many revolutions before flying back to infinity.

In the elliptical orbits, for the Newtonian case the frequency of oscillation in r and ϕ match exactly (this is true only for $1/r$ and r^2 potentials – see, e.g. Classical Mechanics by Goldstein), so the orbit is closed. In the Schwarzschild case this is not so, by the time the particle turns around and reaches the same r, the ϕ coordinate has advanced more than 2π. This causes the orbit to precess around, as seen in the so-called perihelion advance of the planet Mercury by about 43 arcseconds per century.

Photon orbits

Continuing on the theme of trajectories in Schwarzschild spacetime, let us now examine the propagation of light (or other massless ultrarelativistic particles). One feature of light propagation we have come across already – the Gravitational Redshift. In Schwarzschild spacetime the gravitational redshift is given by

$$\nu_\infty = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \nu$$

Equations for the classical light trajectories can be obtained from those derived for particle trajectories, by setting $m \to 0$, $\bar{E} \to \infty$, $\bar{a} \to \infty$ and by noting that $\bar{a}/\bar{E} \to \bar{l}$, where $\bar{l} = l/r_g$ is the normalised impact parameter at infinity.

$$\left(\frac{d\bar{x}}{d\tau}\right)^2 = 1 - \frac{\bar{l}^2}{\bar{r}^2} + \frac{\bar{l}^2}{\bar{r}^3}$$

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{\bar{l}^2}{\bar{r}^4}\left(1 - \frac{1}{\bar{r}}\right)$$

Since $d\bar{r} = (1 - 1/\bar{r})d\bar{x}$ and $\bar{r} = r/r_g$, one can write the following orbit equation:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{l^2}\left(1 - \frac{l^2}{r^2} + r_g\frac{l^2}{r^3}\right)$$

which describes the trajectory of light. In flat space the third term in the parentheses is absent, and one obtains a straight line as the solution. With the third term included, the light trajectories are curved. This demonstrates that gravity bends light.

The nature of the light trajectories can be analysed from the above equations. If a light ray, coming from infinity, has to escape back to infinity, $(d\bar{x}/d\tau)^2$ must always be positive. This is possible only if the impact parameter at infinity l is larger than $l_{\rm min} = 3\sqrt{3}r_g/2$. Rays with smaller l will be gravitationally captured by the central object (black hole). Rays with l very close to, but higher than $l_{\rm min}$, will go round the hole several times at $r \approx 3r_g/2$ before escaping back to infinity. Light trajectories in Schwarzschild spacetime with a few different values of l are plotted in figure 1. As is clear, the amount of deflection the light trajectory suffers

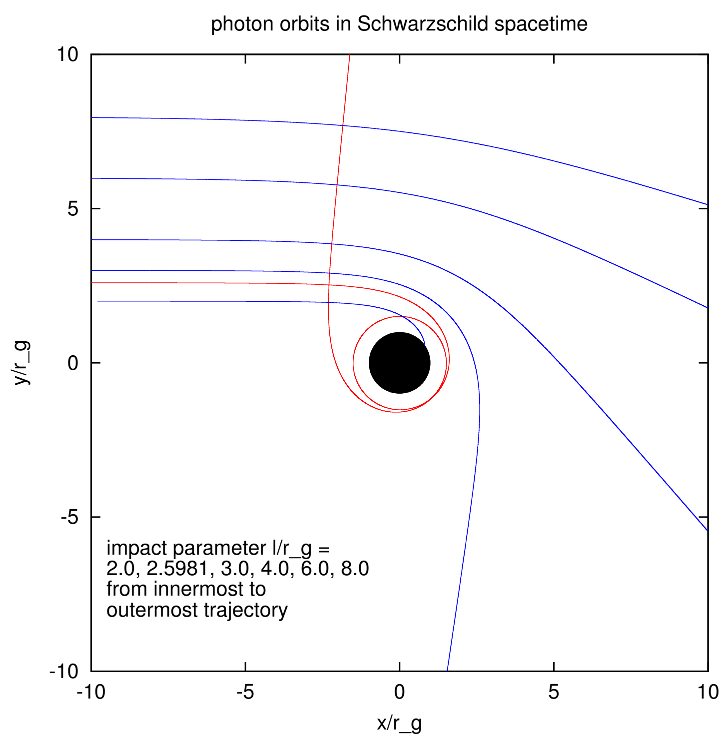

Figure 2: Photon orbits around a Schwarzschild Black Hole. The red trajectory has an impact parameter very close to $3\sqrt{3}r_g$, which makes it go round the central mass more than once before escaping to infinity.

reduces as l increases. Far away from the gravitational radius, the amount of deflection reaches the asymptotic value

$$\Delta\phi = \frac{4GM}{c^2 l}$$

The approach of the deflection to this asymptotic limit is displayed in figure 2. For a ray grazing the surface of the Sun, the amount of deflection works out to be 1.75 seconds of arc, and the observation of this deflection was used as one of the first tests of General Relativity.

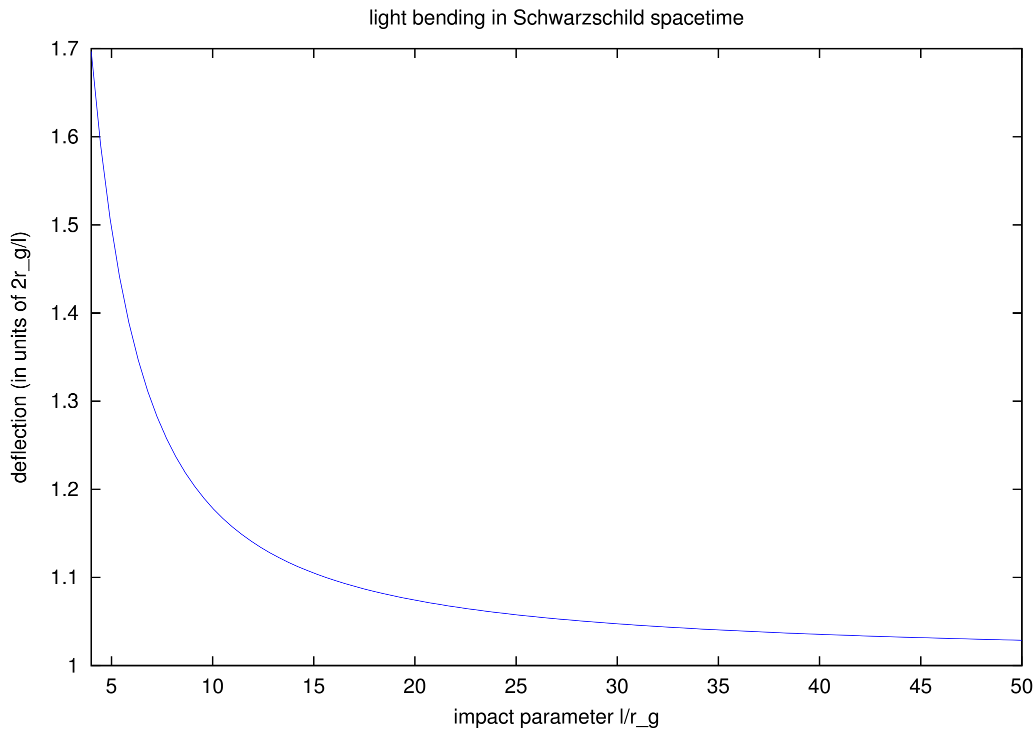

Figure 3: Amount of light bending in Schwarzschild geometry as a function of the impact parameter l at infinity. The bending is normalised to the asymptotic value of $4GM/c^2l$

Light bending by large masses in the Universe, including galaxies and clusters of galaxies, leads to the observed phenomenon of Gravitational Lensing, where a distorted and magnified image of a source in the background is seen by the observer. See figure 3 for such an example. Depending on the observing geometry, the distorted image may sometimes appear as multiple images of the same source. Modelling of the image pattern gives us a way to infer the masses of the galaxies or clusters acting as the lenses.

Lensing by individual stars also reveal themselves in a different way. Since the angle of deflection by a stellar mass object is small, it is usually not possible to observe the distorted images by existing techniques. But the total light received from the source depends on the net magnification. If the star is moving with respect to the source, the observed magnification changes with time due to the change in viewing geometry, and this causes the observed intensity from the source to vary in a predictable manner. This effect is called “microlensing” and is being actively used to search for sub-stellar mass objects.

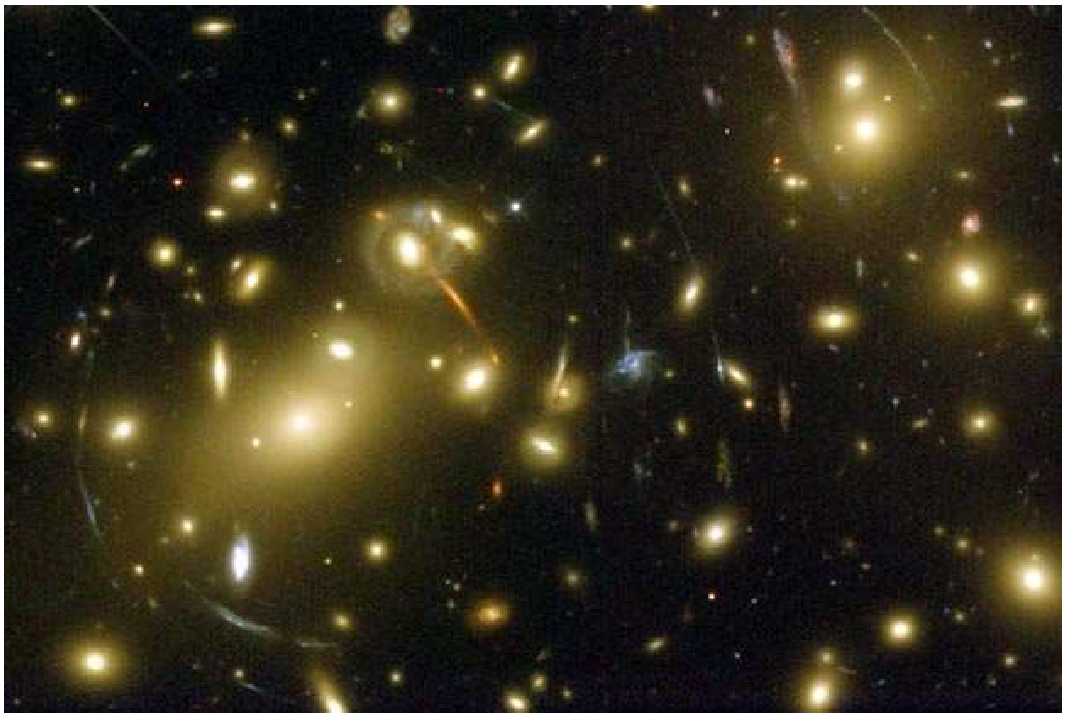

Figure 4: This picture of galaxy cluster Abell 2218 by the Hubble Space Telescope shows distorted images of background galaxies, as a result of gravitational lensing. Picture courtesy Andrew Fruchter et al. at the Space Telescope Science Institute.

Distributed matter

So far we have considered gravitational field of spherically symmetric distribution of masses, as experienced from outside the mass distribution – for which only the total mass of the system is important. In case of distributed masses, the gravitational field has to be obtained from the Einstein equation

$$R_{ik} - \frac{1}{2} g_{ik} R - g_{ik}\Lambda = \frac{8\pi G}{c^4} T_{ik}$$

where the Ricci tensor R_{ik} and the scalar curvature R, constructed out of the metric coefficients g_{ij}, are measures of the curvature of spacetime. The quantity T_{ik} is the energy-momentum tensor of the distributed matter (also called the *stress tensor*). In astrophysical situations, we will deal mainly with gaseous material. In a frame comoving with the gas, the stress tensor is diagonal, and has components:

$$T_{ij} = \mathrm{dia}[\epsilon, P, P, P]$$

where ϵ is the energy density of the gas and P is its pressure.

The term $g_{ik}\Lambda$ corresponds to the energy-momentum tensor of the vacuum. This amounts to an energy density of the vacuum $\epsilon_\Lambda = c^4\Lambda/8\pi G$ pervading over all space, with a corresponding pressure $P_\Lambda = -\epsilon_\Lambda$ (note the negative pressure). The above energy density is equivalent to a mass density $\rho_\Lambda = \epsilon_\Lambda/c^2$. This vacuum mass density is so small that in all contexts other than Cosmology the presence of this term in Einstein's equation can be neglected. Indeed the measurement of Λ comes from Cosmology, and it is inferred to be about 10^{-52} m^{-2}, giving $\rho_\Lambda \approx 10^{-26}$ kg/m^3.

Equations of motion of matter can be written using the conservation of energy and momentum:

$$\frac{\partial T_i^k}{\partial x^k} + \Gamma_{lk}^k T_i^l - \Gamma_{ik}^l T_l^k = 0$$

which, for a gas in a comoving frame yield the following.
For $i = 0$:

$$dU + PdV = 0$$

where V is the volume and $U = \epsilon V$ is the total energy of the system. This is equivalent to the First law of Thermodynamics.

The other components of the equation give:

$$\frac{\partial P}{\partial x^\alpha} - \left(\frac{g_{0\alpha}}{g_{00}}\right)\left(\frac{\partial P}{\partial x^0}\right) = (\epsilon + P)\frac{F_\alpha}{c^2}; \quad \alpha = 1, 2, 3$$

where F_α is the respective component of acceleration (force per unit mass).

For matter at rest ($\partial P/\partial x^0 = 0$), one gets equations of Hydrostatic equilibrium:

$$\frac{\partial P}{\partial x^\alpha} = \frac{\epsilon + P}{c^2} F_\alpha = (\rho + \frac{P}{c^2}) F_\alpha$$

In the non-relativistic limit ($\epsilon \gg P$) the right hand side approximates to just ρF_α. We will need to use the equations of hydrostatic equilibrium in the study of the structure of several astronomical objects including stars.

Gravitational Radiation

We will conclude this discussion of gravity with a brief mention of gravitational radiation. General Relativity predicts the existence of gravitational radiation from moving masses, analogous to electromagnetic radiation from moving charges. There is, however, an important difference. As we know, electromagnetic radiation, in the leading order, comes from varying electric dipole moment and the power radiated is given by the Larmor formula:

$$-\frac{dE}{dt} = \frac{2}{3c^3}(\ddot{D})^2$$

where D is the electric dipole moment. In case of gravitational radiation, the leading order emission comes from varying quadrupole moment of mass distribution:

$$-\frac{dE}{dt} = \left(\frac{G}{45c^5}\right)\left(\frac{\partial^3 K_{\alpha\beta}}{\partial t^3}\right)^2$$

where

$$K_{\alpha\beta} = \int \rho(3x^\alpha x^\beta - \delta^\beta_\alpha x^\gamma x^\gamma) dV$$

is the quadrupole moment of the mass distribution.

The application of this to binary stars is of major astrophysical importance. A binary star is a two-body system, mainly in the weak-field limit and hence well described by Keplerian orbit in the first order. The relative separation r between the component masses m_1 and m_2 is given by

$$r = \frac{a(1-e^2)}{1+e\cos\psi}$$

where a is the semi-major axis of the orbit and ψ is the polar angle. The expression of $-dE/dt$ above can be evaluated at any point of this orbit. Averaged over the entire orbit, this yields

$$L_{\rm grav} = -\left(\frac{dE}{dt}\right)_{\rm av} = \frac{32}{5}\frac{G^4}{c^5}m_1^2 m_2^2(m_1+m_2)\frac{1}{a^5}f(e)$$

with

$$f(e) = \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}$$

This radiation produces change in both the orbital separation and the eccentricity. Since the total energy of the orbit is

$$E = -\frac{1}{2}\frac{Gm_1m_2}{a}$$

the orbit shrinks at the rate

$$\frac{da}{dt} = \frac{2a^2}{Gm_1m_2}\frac{dE}{dt}$$

while the eccentricity reduces at the rate

$$\frac{de}{dt} = (1-e)\frac{1}{a}\frac{da}{dt}$$

Under the influence of this gravitational radiation, the two components of a binary will eventually merge, given sufficient time. The shrinking of orbit due to gravitational radiation has already been observed in at least one binary system containing two neutron stars. However gravitational radiation has so far not been directly detected.

We are now witnessing the commissioning of a number of sensitive detectors in several places around the world which have been built to detect gravitational radiation from such inspiralling binaries. In the next decade, it is planned to launch a Laser Interferometric Space Array for gravitational wave detection, which will be much more sensitive than the ground based detectors currently being commissioned.