Reference: https://casper.berkeley.edu/astrobaki/index.php/Cosmology
I've simply worked through the equations in lectures 1 and 2 here.
I acknowledge Aaron Parsons and at UC Berkely and the resources available online at astrobaki

→ $r = a(t)\, x$

$$\dot{r} = \dot{a}x + a\dot{x} = \underbrace{\frac{\dot{a}}{a}}_{H} r + \underbrace{a\dot{x}}_{= v_p}$$

H → Hubble

v_p → Peculiar

→ Adiabatic — no heat is added or removed from the system. — does not necessarily mean that the temperature is unchanged [isothermal]

→ So if expansion occurs → the system/gas will cool. So if $V \uparrow \rightarrow T \downarrow \rightarrow$ adiabatic.
In isothermal, you add heat / energy to keep the T the same

→ Friedmann eqn - eqn for motion for a(t)

$$M = \frac{4\pi}{3} a^3 \rho$$

Energy / mass / length

$$E = \underbrace{\frac{1}{2}\dot{a}^2}_{KE} - \underbrace{\frac{G \cdot M}{a}}_{PE} = \frac{1}{2}\dot{a}^2 - \frac{4\pi G \cdot \rho a^2}{3}$$

$$k = -\frac{2E}{c^2} \rightarrow \text{curvature}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2}$$

↳ 1st

$$E = \frac{1}{2}(\dot{a})^2 - \frac{GM}{a} = \frac{1}{2}\dot{a}^2 - G \cdot \frac{4\pi}{3}\frac{\rho a^{\cancel{3}2}}{\cancel{a}}$$

$$M = \frac{4\pi}{3}\rho a^3 \rightarrow \rho = \left(\frac{4\pi}{3}\frac{a^3}{M}\right)^{-1}$$

$$\rho = \frac{3}{4\pi}\frac{M}{a^3}$$

$$E = \frac{1}{2}\dot{a}^2 - G\frac{4\pi}{3}\rho a^2$$

$$k \equiv \frac{E}{c^2}$$

$$\frac{E}{a^2} = \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{4\pi}{3}G\rho$$

$$\frac{E}{c^2}\cdot\frac{c^2}{a^2} = \frac{1}{2}H^2 - \frac{4\pi}{3}G\rho$$

$$H^2 = -\frac{\cancel{2}kc^2}{a^2} + \frac{8\pi}{3}G\rho.$$

$$\boxed{H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \rightarrow 1^{st} \text{ Friedmann eqn.}}$$

↳ conservation of E.

$\Delta S = 0 \rightarrow$ change in entropy zero.

$\Delta Q = T\,\Delta S \rightarrow$ change in heat

$\Delta S \rightarrow 0 \Rightarrow \Delta Q \rightarrow 0.$

If U is the internal energy & P the pressure

$$\frac{dU}{dt} = -P\frac{dV}{dt} \rightarrow$$ +ve pressure must lose energy as vol. expands.

In an expanding Universe

$\cancel{E}$ $U = \frac{E}{V}.V = \rho a^3$

→ energy density

$P \rightarrow$ pressure of Photon gas

|

① In the universe following 1 law of TD adiabatic expansion

$$dQ = 0 = dU + P\,dV$$

$dQ \rightarrow$ total heat/energy

$dU \rightarrow$ int energy

$P \rightarrow$ pressure

$dV \rightarrow$ vol

② energy density $\rho = \dfrac{U}{V}$

$$d\rho = d\left(\frac{U}{V}\right) = \frac{dU}{V} - dV \cdot \frac{U}{V^2}$$

$$= \frac{dV}{V}\left(\frac{dU}{dV} - \frac{\cancel{dV} \cdot U}{V}\right)$$

$$= \frac{dV}{V}\left(-P - \rho\right)$$

$\dfrac{U}{V}$ = energy density ρ

$$d\rho = -(P + \rho)\frac{dV}{V}$$

$\dfrac{dU}{dV}$ = pressure?

$V \propto a^3$ $\qquad dV \propto 3a^2\,da$

$$\frac{dV}{V} = \frac{3\cancel{a^2}\,da}{a^{\cancel{3}}} = 3\frac{da}{a}$$

$k = \frac{E}{c^2}$

$$d\rho = -3(p+\rho)\frac{da}{a}$$

$$\boxed{\dot{\rho} = -3(p+\rho)\frac{\dot{a}}{a} \rightarrow \text{Friedmann eqn}}$$

↳ Temperature loss due to ~~energy~~ adiabatic expansion of the Universe.

③. $\frac{d}{dt}\left[1^{st} \text{ Friedmann eqn}\right]$

$$\frac{d}{dt}\left[H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}\right]$$

$$\frac{d}{dt}\left[\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}\right]$$

$$\cancel{2}\cancel{\dot{a}}\dot{a} \quad \cancel{2}\left(\frac{\cancel{2}\dot{a}}{a}\right) \quad \frac{d}{dt}\left(\frac{\dot{a}}{a}\right)^2 \quad 2\left(\frac{\dot{a}}{a}\right)\left[\frac{\ddot{a}}{a} - \frac{1}{2}\frac{\dot{a}}{a^2}\right]$$

$$\frac{2\dot{a}\ddot{a}}{a^2} - \frac{\dot{a}^2}{a^3},$$

$$\frac{d}{dt}\left[\dot{a}^2 = \frac{8\pi G\rho a^2}{3} - kc^2\right]$$

$$\cancel{2\dot{a}\ddot{a}} = \cancel{3A}\ \frac{8\pi G\rho\, 2a\dot{a}}{3}$$

$2\dot{a}\ddot{a}$

$$\frac{d}{dt}\left[(\dot{a})^2 = \frac{8\pi}{3} G\rho a^2\right].$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\frac{d}{dt}(\rho a^2)$$

$$= \frac{8\pi G}{3}\left[\dot{\rho} a^2 + 2a\dot{a}\rho\right]$$

$$= \frac{8\pi G}{3}\left[-3(p+\rho)\frac{\dot{a}}{\cancel{a}}\cdot a^{\cancel{2}} + 2a\dot{a}\rho\right]$$

$$\cancel{2}\dot{a}\ddot{a} = \frac{4\cancel{8}\pi G}{3}\left[-3p\dot{a}a - \rho\dot{a}a\right].$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho + 3p].}$$

↳ Friedmann's third eqn.

Friedmann's eqns

Friedmann's eqns. summarized

$$k = \frac{-E}{c^2}$$

① $$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2}$$

Cons of E

↳ energy conserved.

From I law of TD

② $$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} = -(\rho + p)\frac{\dot{V}}{V}$$

Acceleration of scale factor

③ $$\frac{\ddot{a}}{a} = -\frac{4\pi}{G}(\rho + 3p)$$

$t \propto (1+z)^{-3/2}$

Invoke equation of state w to relate pressure & density

$$\boxed{P = w \rho (c^2)}$$

usually & $c = 1$.

$\therefore$ from F(II) $\quad \dot{\rho} = -3(P + \rho)\frac{\dot{a}}{a}$.

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \rho (1 + w)$$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}.$$

$$\ln \rho = \ln a^{-3(1+w)}$$

$$\boxed{\rho = a^{-3(1+w)}.}$$

1) Pressureless $\rightarrow$ $w = 0$.

For the case of matter (non-relativistic)

$$\rho \propto a^{-3}$$

which is okay as $V \propto a^3$

2) Relativistic $\Rightarrow$ $w = 1/3$.

(photons, bosons, neutrinos?)

$$\rho = a^{-3(1+1/3)}$$

$$\rho \propto a^{-4}$$

3) Dark energy

$w = -1$.

$\rho \propto a^0 \rightarrow$ constant $\bar{c}$ time

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Critical Density:

Critical density

$$H_0^2 = \frac{8\pi G\rho}{3} \quad (k=0)$$

$$\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$$

$$\rho = \underbrace{\left(H_0^2 + \frac{kc^2}{a^2}\right)}_{\text{const}}$$

$$= \frac{30000}{8\pi G} h^2$$

ρ k

Conditions for :
Open, closed, flat Universe

$$\boxed{\Omega = \frac{\rho}{\rho_c}}$$

$$\Omega \begin{cases} < 1 & k < 0 \rightarrow \text{open} \\ = 1 & \text{— } \cancel{\text{critical}}\ \hookrightarrow \text{flat} \quad k = 0 \\ > 1 & \hookrightarrow k > 0 \rightarrow \text{crunch closed} \end{cases}$$

$$\Omega = \sum_i \Omega_i \begin{cases} r \ \text{radiation} \\ M \ \text{matter} \rightarrow \text{dark \& luminous} \\ \quad b \rightarrow \text{baryons} \\ \nu \rightarrow \text{neutrinos} \\ \Lambda \rightarrow \text{dark E.} \end{cases}$$

Evolution of Hubble constant with redshift / scale factor

Evolution of $H(t)$

$$H_0^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a_0^2}$$

$c = 1; \quad a_0 = 1.$ $\qquad \frac{\rho_0}{\rho_{crit}} = \Omega_0$

$$k = \frac{8\pi G\rho}{3} - H_0^2 \qquad \frac{3H_0^2}{8\pi G} \cdot \frac{1}{H_0^2}$$

$$= H_0^2(\Omega_0 - 1) \longrightarrow (1)$$

$$H^2 = \frac{8\pi G \rho}{3} - \frac{kc^2}{a^2}$$ → at some other time

$\rho \propto a^{-3(1+w)}$ EoS ρ $\rho_0 \cdot a^{-3}$ $\rho \propto V^{-1}$ V

$c = 1$

$$\frac{k}{a^2} = \frac{8\pi G}{3H_0^2} \cdot H_0^2 \cdot \rho - H^2 \quad \text{②}$$

From a^2(2) = (1)

$$a^2\left(\frac{\rho}{\rho_{crit,0}} \cdot H_0^2 - H^2\right) = H_0^2(\Omega_0 - 1)$$

Relation between \rho and scale factors on next page

$$a^2 H^2 = \left(\frac{\rho_0 \cdot a^{-3(1+w)}}{\rho_{crit,0}} \cdot H_0^2\right) a^2 - H_0^2(\Omega_0 - 1)$$

$$\boxed{H^2 = H_0^2\left(\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}\right)}$$

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Quick recap of scale factor:

l

$$\frac{l}{l_0} = \frac{a}{\cancel{a_0}\ 1}$$

ignore

$$l = a\, l_0$$

l_0

Subscript 0 denotes z=0, i.e. present values

$$l^3 = a^3 l_0^3$$
$$V = a^3 V_0$$

$$\rho = a^{-3} \rho_0$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{k(c^2 \Rightarrow 1)}{a^2}$$

For Einstein-de Sitter Universe $k=0$; $\rho \propto a^{-3(1+w)}$

$$\left(\frac{\dot a}{a}\right)^2 \propto \frac{8\pi G\, a^{-3(1+w)}}{3}$$

$$\left(\frac{\dot a}{a}\right)^2 \propto a^{-3(1+w)}$$

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$$\frac{\dot a}{a} \propto a^{-\frac{3}{2}(1+w)}.$$

$$\dot a \propto a^{-\frac{3}{2}(1+w)+1}.$$

$$\frac{da}{dt} \propto a^{-\frac{3}{2}(1+w)+1}$$

$$a^{-1} \cdot a^{+\frac{3}{2}(1+w)}\, da \propto dt.$$

$$\int \frac{a^{3/2(1+w)}}{a}\, da \propto \int dt$$

$$\frac{d}{da}\left[\ x\ \right] = \frac{a^{3/2(1+w)}}{a}$$

$$\frac{d}{da}\left[a^{3/2(1+w)}\right] = \left[\frac{3}{2}(1+w)\right] a^{\frac{3}{2}(1+w)-1}$$

$$\propto \frac{a^{3/2(1+w)}}{a}. =$$

$$x = a^{3/2(1+w)}$$

$$\therefore \quad a^{3/2(1+w)} \propto t$$

$$\boxed{a \propto t^{\frac{2}{3(1+w)}}}$$

$$a \propto t^{\frac{2}{3(1+w)}}$$

① $w = 0 \longrightarrow \Omega = \Omega_m$ $\quad P = 0$

$$\boxed{a \propto t^{2/3}. \quad \text{Matter dominated}}$$

② $w = 1/3 \longrightarrow \Omega = \Omega_r$ $\quad [P_{rad} = 1/3\, \rho_{rad}]$

$$\boxed{a \propto t^{\frac{2 \cdot 3}{3(4)}} \propto \underline{\underline{t^{1/2}}} \quad \text{Radiation dominated}}$$

③ $w = -1$

$a \propto$ ~~x~~

$$\boxed{a(t) \propto t^{\frac{1}{0}}. \quad \text{wrong} \quad \times}$$

$$\int \frac{1}{a} a^{\frac{3}{2}(1+w)} da \propto \int dt$$

$w = -1$

$$\int \frac{1}{a} da \propto \int dt$$

$$\ln a \propto t$$

$$a \propto e^{t}$$

Dark Energy Dominated