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High-energy tail of the velocity distribution of driven inelastic Maxwell gases

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Abstract - A model of homogeneously driven dissipative system, consisting of a collection of \( N \) particles that are characterized by only their velocities, is considered. Adopting a discrete time dynamics, at each time step, a pair of velocities is randomly selected. They undergo inelastic collision with probability \( p \). With probability \((1 - p)\), energy of the system is changed by changing the velocities of both the particles independently according to \( v \to -r_w v + \eta \), where \( \eta \) is a Gaussian noise drawn independently for each particle as well as at each time steps. For the case \( r_w = -1 \), although the energy of the system seems to saturate (indicating a steady state) after time steps of \( O(N) \), it grows linearly with time after time steps of \( O(N^2) \), indicating the absence of an eventual steady state. For \(-1 < r_w < 1\), the system reaches a steady state, where the average energy per particle and the correlation of velocities are obtained exactly. In the thermodynamic limit of large \( N \), an exact equation is obtained for the moment generating function. In the limit of nearly elastic collisions and weak energy injection, the velocity distribution is shown to be a Gaussian. Otherwise, for \(|r_w| < 1\), the high-energy tail of the velocity distribution is Gaussian, with a different variance, while for \( r_w = +1 \) the velocity distribution has an exponential tail.

The steady-state velocity distributions measured in experiments show deviation from the Maxwellian statistics as well as approach to Gaussian distribution, depending on the experimental conditions \([1–6]\). Based on the kinetic theory, an analysis by van Noije and Ernst \([7]\) for a uniformly heated inelastic hard sphere gas, predicts a non-Maxwellian tail \( P(v) \sim \exp(-A|v|^\alpha) \) with \( \alpha = 3/2 \), of the velocity distribution. Barrat et al. \([8]\) find \( \alpha = 3 \) in the limit of vanishing inelasticity (see also \([9]\)). On the other hand, the numerical study of a two-dimensional driven inelastic gas \([10]\) shows a wide range of \( \alpha (\alpha < 2) \) depending on the various system parameters, instead of a universal value. A one-dimensional model of granular system consisting of Brownian particles subject to inelastic mutual collisions, shows a crossover from Gaussian to non-Gaussian distribution, as the ratio of the Brownian...
Gaussian velocity distribution has been obtained, while in the results for \( e(n) \), the value is shown by the (magenta) dotted line. (b) Simulation results for (red circles) as well as analytical (black solid line) results for (red circles) and \( \Sigma(n) \) (orange squares) of \( \epsilon = 5000 \) particles system driven by adding uncorrelated white noise, \( \eta = 1/4 \). Analytical results are shown by (black) solid and (blue) dashed lines, respectively. The pseudo steady-state value is shown by the (magenta) dotted line. (b) Simulation results for (red circles) and \( \Sigma(n) \) (orange squares) with \( N = 1000 \), \( p = 1/2 \) and \( \epsilon = 1/4 \), for \( \delta = 1 \) case. (c) Simulation (red circles) as well as analytical (black solid line) results for (red circles) and \( \Sigma(n) \) (orange squares) of \( \epsilon = 4/9 \) for \( N = 5000 \), with the dynamics given by eq. (2), with \( p = 1/2, \epsilon = 1/4, \) and \( \omega_{r} = 1/2 \). The steady-state value is shown by the (magenta) dotted line. (d) Simulation (red circles) as well as analytical (black solid line) results for (red circles) and \( \Sigma(n) \) (orange squares) of \( \epsilon = 4/9 \) for \( N = 5000 \), with the dynamics given by eq. (2), with \( p = 1/2, \epsilon = 1/4, \) and \( \omega_{r} = 1/2 \). The steady-state value is shown by the (magenta) dotted line.

Fig. 1: (Color online) (a) Simulation results for the evolution of the energy per particle \( e(n) \) (red circles) and the velocity correlation \( \Sigma(n) \) (orange squares) of \( N = 5000 \) particles system driven by adding uncorrelated white noise \( \omega_{r} = -1 \), for \( p = 1/2 \) and \( \epsilon = 1/4 \). Analytical results are shown by (black) solid and (blue) dashed lines, respectively. The \( \text{pseudo} \) steady-state value is shown by the (magenta) dotted line. (b) Simulation results for (red circles) and \( \Sigma(n) \) (orange squares) with \( N = 1000 \), \( p = 1/2 \) and \( \epsilon = 1/4 \), for \( \delta = 1 \) case. (c) Simulation (red circles) as well as analytical (black solid line) results for (red circles) and \( \Sigma(n) \) (orange squares) of \( \epsilon = 5000 \), with the dynamics given by eq. (2), with \( p = 1/2, \epsilon = 1/4, \) and \( \omega_{r} = 1/2 \). The steady-state value is shown by the (magenta) dotted line. (d) Simulation (red circles) as well as analytical (black solid line) results for (red circles) and \( \Sigma(n) \) (orange squares) of \( \epsilon = 4/9 \) for \( N = 5000 \), with the dynamics given by eq. (2), with \( p = 1/2, \epsilon = 1/4, \) and \( \omega_{r} = 1/2 \). The steady-state value is shown by the (magenta) dotted line.

Relaxation time to the mean collision time is increased [11]. A model of a driven dissipative system was studied in [12] where the energy moments could be computed exactly.

Kinetic theory approach to homogeneous granular gases involves setting up the Boltzmann equation for the single-particle velocity distribution, under the molecular chaos hypothesis, which replaces the two-particles velocity distribution function by the product of two single-particle probability density functions (PDFs) \( P(v_{1}, v_{2}) = P(v_{1})P(v_{2}) \). For hard spheres, the collision rate is proportional to \( |v_{1} - v_{2}|^{6} \) with \( \delta = 1 \). In the absence of external driving it has been shown that the velocity distribution has an exponential tail \( P(v) \sim \exp(-|v|/v_{0}(t)) \) where \( v_{0}(t) \) is the time-dependent thermal speed [13]. In the driven case, uniform heating is modeled by adding a diffusive term \( \Delta \alpha^{2}P(v) \) to the Boltzmann equation. Ben-Naim and Krapivsky introduced a simpler model, called inelastic Maxwell model [14], where the collision rate is independent of the velocities of the colliding particles, i.e., \( \delta = 0 \). In this model, an exponential tail \( P(v) \sim \exp(-A|v|) \) for the velocity distribution has been obtained [15–17], while in the limit of vanishing inelasticity, the distribution becomes Gaussian [18].

The inelastic Maxwell model is perhaps the simplest model of granular gases. The modeling of velocity change due to an external forcing (heating) by the uncorrelated white noise, \( \eta \), (which corresponds to the diffusion term in the Boltzmann equation) injects energy on average, to the system. However, as it turns out, this energy input cannot be balanced by the energy dissipation due to inelastic collisions, and eventually, the average energy of the system increases linearly with time and hence there is no steady state (see fig. 1(a)). This is true even for \( \delta > 0 \), as seen from fig. 1(b). This is because the inter-particle collision conserves total momentum while the driving noise causes the total momentum to perform a random walk—hence the average energy of the system grows linearly with time. One usually justifies this model of heating by moving to the center of mass velocity frame [14,19]. However, it is not the experimental situation and hence is unsatisfactory. In fact, as we show below, the external driving is better modeled by a Ornstein-Uhlenbeck process, if one assumes that the external forcing is due to collisions with a wall with a random velocity. While, similar forcing mechanism has been considered earlier for inelastic gases [11,17,20,21], the form of the high-energy tail has not been obtained. In this letter, we obtain several exact results, including the high-energy tail of the velocity distribution.

Here for simplicity, we consider the dynamics in discrete time steps. It is straightforward to obtain the continuum version by taking appropriate limits [22]. Following ref. [14], ignoring the spatial structure, we consider a collection of \( N \) identical particles that are characterized by their velocities \( v_{i} \), where \( i = 1, 2, \ldots, N \). The initial velocities are chosen independently from a Gaussian distribution. At each time step, two particles are chosen at random, and with probability \( p \) they undergo inelastic collision and with probability \( 1 - p \) they are subjected to independent external forcing. The two-body inelastic collisions change the velocities of the selected pair from \( (v_{1}, v_{2}) \) to \( (v_{1}', v_{2}') \) according to

\[
\begin{align*}
v_{1}' &= \epsilon v_{i} + (1 - \epsilon)v_{j}, \\
v_{2}' &= (1 - \epsilon)v_{i} + \epsilon v_{j}.
\end{align*}
\]

Here \( \epsilon = (1 - r)/2 \), with \( r \) being the coefficient of restitution defined by \( (v_{1}' - v_{2}') = -r(v_{1} - v_{2}) \). The elastic collision corresponds to \( r = 1 \). Although \( r = 0 \) corresponds to the completely inelastic case, in this model the particles do not stick to each other and merely possess the same velocity after collision. Therefore, collision conserves both the number of particles and total momentum \( (v_{1}' + v_{2}') = v_{1} + v_{2} \). In fact, as a model of a dissipative system, we can take \( \epsilon \in (0, 1) \).

As in ref. [14], we first apply the external forcing by adding uncorrelated white noises to the velocities \( (v' = v + \eta) \) of the selected pair independently. In this case, starting from an initial state of uncorrelated velocities, we find that [22] for large \( N \), after time steps of \( O(N) \) the
system reaches a steady-state-like state where the average energy per particle saturates to a fixed value (fig. 1(a)). Although, correlation between velocities builds up over time through collisions, it is $O(1/N)$ in this pseudo steady state and therefore remains negligible. Consequently, the time through collisions, it is $\frac{2p(1-\epsilon)}{N}$

\[ R = \begin{bmatrix}
1 & \frac{4p(1-\epsilon) + 2(1-p)(1-log_{\Sigma})}{N} \\
\frac{8p(1-\epsilon)}{N(N-1)} & \frac{2p(1-\epsilon)}{N(N-1)}
\end{bmatrix}, \quad \text{and}
\]

\[ e = \frac{(\sigma^2/2)[2\epsilon(1-\epsilon) + \gamma(1-r_w^2) + 2(N-2)\gamma(1+r_w)]}{4\epsilon(1-\epsilon)(1-r_w^2) + \gamma(1-r_w^2)^2 + (N-2)(1+r_w)[4\epsilon(1-\epsilon) + 2\gamma(1-r_w^2)]}, \quad \text{(7)}
\]

\[ \Sigma = \frac{2\sigma^2(1-\epsilon)}{4\epsilon(1-\epsilon)(1-r_w^2) + \gamma(1-r_w^2)^2 + (N-2)(1+r_w)[4\epsilon(1-\epsilon) + 2\gamma(1-r_w^2)]}, \quad \text{(8)}
\]

the $n$-th time step are defined as

\[ e(n) = \frac{1}{2N} \sum_{i=1}^{N} \langle v_i^2(n) \rangle, \quad \text{(3)}
\]

\[ \Sigma(n) = \frac{1}{N(N-1)} \sum_{i \neq j} \langle v_i(n)v_j(n) \rangle, \quad \text{(4)}
\]

respectively. It turns out that they satisfy the following exact recursion relation:

\[ X_n = RX_{n-1} + C, \quad \text{(5)}
\]

where $X_n$, $R$, and $C$ are given by

\[ \text{see eq. (6) above} \]

Equation (5) has the exact solution $X_n = R^n X_0 + \sum_{i=0}^{n-1} R^i C$, with the initial conditions $X_0 = [e, 0]^T$. The two eigenvalues of $R$ can be evaluated explicitly. For $r_w = -1$, the eigenvalues are 1 and $1 - p(1 - r^2)/(N - 1)$. The unit eigenvalue leads to an eventual linear increase of $X_n$. For $r_w \neq -1$, both eigenvalues have absolute values less than unity, and hence in the $n \rightarrow \infty$ limit, $X_n \rightarrow X_\infty$, independent of $n$, as seen in fig. 1(c) and fig. 1(d). Thus, the system reaches a steady state. The energy and correlations in the steady state are given exactly by

\[ \text{see eqs. (7) and (8) above} \]

respectively, where $\gamma = (1-p)/p$, is the ratio of the injection to the collision rate. In the large $N$ limit we get

\[ e = \frac{\gamma \sigma^2}{4\epsilon(1-\epsilon) + 2\gamma(1-r_w^2)} + O(N^{-1}) \quad \text{(9)}
\]

and $\Sigma = O(N^{-1})$. Therefore, in the steady state, in the thermodynamic limit $N \rightarrow \infty$, the correlations between the velocities vanish. In this limit, it is easy to show that, the moment generating function $Z(\lambda) = \exp(-\lambda v)$ of the steady-state velocities satisfies the equation

\[ Z(\lambda) = pZ(\epsilon_\lambda)Z([1-\epsilon_\lambda]) + (1-p)Z(r_\lambda) f(\lambda), \quad \text{(10)}
\]

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where \( f(\lambda) = \exp(\lambda^2 \sigma^2/2) \) and we have used the fact that \( Z(\lambda) = Z(\lambda) \) for even distribution. Note that \( Z(\lambda) = 1 + e^{\lambda^2} + \cdots \) as \( \lambda \to 0 \).

In the near-elastic and weak energy injection limit: \( \epsilon \to 0, r_w = (1 - \theta) \to 1, \sigma \to 0 \), while keeping \( \sigma^2/\epsilon \) and \( \theta/\epsilon \) fixed, using the Taylor expansion in eq. (10) we get

\[
\frac{dZ}{d\lambda} = \lambda \Delta^2 Z(\lambda), \quad \text{where} \quad \Delta^2 = \frac{\gamma \sigma^2}{2[1 + \gamma \theta/\epsilon]}.
\]

Note from eq. (9) that \( \langle v^2 \rangle \to \Delta^2 \) in this limit. Evidently, the solution of eq. (11) is \( Z(\lambda) = \exp(\lambda^2 \Delta^2/2) \), which implies the Gaussian velocity distribution

\[
P(v) = \frac{1}{\sqrt{2 \pi \Delta^2}} \exp \left( -\frac{v^2}{2 \Delta^2} \right).
\]

Figure 2 shows a comparison of this result with numerical simulation.

In general, we are not able to find the exact solution of eq. (10). For the case \( r_w = +1 \), we have

\[
Z(\lambda) = \left[ 1 - (1 - p)f(\lambda) \right]^{-1} p Z(\epsilon \lambda) Z(1 - \epsilon \lambda),
\]

which can be solved by iteration [22]. The tail of the velocity distribution is determined by the pole closest to the origin \( \lambda_0 = \pm \sqrt{-2 \ln(1 - p)/\sigma} \), which comes from the factor \( 1 - (1 - p) f(\lambda) \) in eq. (13). This gives rise to exponential tails \( P(v) \sim A(\epsilon) \exp(-\lambda_0 ||v||) \), where the prefactor \( A(\epsilon) \) is known explicitly. A comparison with the numerical simulation gives very good agreement as shown in fig. 3. We note that eq. (10) is symmetric for \( r_w \leftrightarrow -r_w \). However this equation has been derived under the assumption of a steady state. Therefore for the case \( r_w = -1 \), it is valid only in the pseudo-steady state in which case one indeed finds an exponential tail [14–17].

For \( |r_w| < 1 \) it is difficult to obtain the tails accurately from direct numerical simulations, as it would require large number of realizations. Equation (10) can be numerically solved and then inverted numerically to obtain \( P(v) \). It is convenient to use the characteristic function \( Z_c(k) = Z(ik) \). For the special case \( \epsilon = 1/2 \) and \( r_w = 1/2 \), eq. (10) has a simpler form

\[
Z_c(k) = pZ_c^2(k/2) + (1 - p)Z_c(k/2) \exp(-k^2 \sigma^2/2).
\]

This is useful, as to compute \( Z_c \) for any \( k \) one only requires its value at \( k/2 \). This “linear structure”, as opposed to the “tree” in the general case, is useful to efficiently compute \( Z_c(k) \) numerically while using the initial condition \( Z_c(k) = 1 - ek^2 \) for \( k \ll 1 \). Finally, numerically computing the inverse Fourier transform of \( Z_c(k) \) gives the velocity distribution. This is compared with simulation results in fig. 4(c). From the numerical evaluation, we observe the tail \( P(v) \sim \exp(-A|v|^\alpha) \), with \( \alpha \) gradually increasing (but \( < 2 \)) as we go towards higher and higher the velocities.

For \( \alpha > 1 \), the function \( Z(\lambda) \) is analytic. If \( Z(\lambda) \) is known, then the large deviation tail of the velocity distribution can be obtained by the saddle point approximation

\[
P(v) \approx \frac{\exp(\mu(\lambda^*) + \lambda^* v)}{\sqrt{2\pi \mu'(\lambda^*)}},
\]

where \( \mu(\lambda) = \ln Z(\lambda) \) and the saddle point \( \lambda^* (v) \) is implicitly given by the equation \( \mu'(\lambda^*) = -v \). A careful saddle-point analysis [22] shows that the first term on the right hand side of eq. (10) becomes negligible compared to the left hand side, for \( \lambda \) near \( \lambda^* (v) \) for large \( v \). The remaining terms in eq. (10) gives the Gaussian distribution

\[
P(v) \approx \sqrt{\frac{1 - r_w^2}{2\pi \sigma^2}} \exp \left[ -\frac{v^2}{2\sigma^2} (1 - r_w^2) \right].
\]

Therefore, the high-energy tail is governed, not by the inelastic collisions amongst the particles, but by the collisions of the particles with the “wall”.

To verify the above result, we numerically compute \( Z(\lambda) \) from eq. (10) for \( \epsilon = 1/2, r_w = 1/2, \) and \( \sigma = 1 \), for which \( b = 2/3 \). It is clear from fig. 4(a) that \( \mu(\lambda) \sim b\lambda^2 \) for
large $\lambda$. From $\mu(\lambda)$, we also numerically compute $\mu'(\lambda)$ and $\mu''(\lambda)$ (fig. 4(b)). Now, each value of $\lambda$ corresponds to a velocity $v = -\mu'(\lambda)$, whose PDF, under the saddle point approximation, is numerically obtained using

$$P(v = -\mu'(\lambda)) \approx \frac{1}{\sqrt{2\pi \mu''(\lambda)}} \exp[\mu(\lambda) - \lambda \mu'(\lambda)].$$  \hspace{1cm} (17)

Figure 4(c) compares this with the simulation result as well as with the distribution obtained from the exact numerical inverse Fourier transform of $Z_\epsilon(k)$. Computing the moments assuming eq. (16) for all $v$, one gets $\langle v^2 \rangle = (\sigma^2/2)(1 - r_w^2)^{-n}(2n)!/n!$. Certainly this result cannot be valid for small $n$, e.g., compare the $n = 1$ case with 2e form eq. (9). However, for large $n$ one expects this result to agree with the exact result. To compare with the exact result, it is useful to look at the ratio between two successive even moments, $\langle v^{2n}/\langle v^{2n-2}\rangle \rangle \sim 2\sigma^2(1 - r_w^{-1})^{-1}n$ for large $n$. We find that the even moments $M_{2n}$ satisfies the exact recursion relation

$$\left[1 - \epsilon^{2n} - (1 - \epsilon)^{2n} + \gamma(1 - r_w^{2n})\right] M_{2n} =$$
$$\sum_{m=1}^{n-1} \left( \begin{array}{c} 2n \\ 2m \end{array} \right) \epsilon^{2m}(1 - \epsilon)^{2n-2m} M_{2m} M_{2n-2m}$$
$$+ \gamma \sum_{m=0}^{n-1} \left( \begin{array}{c} 2n \\ 2m \end{array} \right) \epsilon^{2m} M_{2m} (2n - 2m)! \left( \frac{\sigma^2}{\gamma} \right)^{n-m},$$  \hspace{1cm} (18)

with $M_0 = 1$. Using this we compute the moments recursively. Figure 5 confirms that $M_{2n}/M_{2n-2} \rightarrow 2\sigma^2 (1 - r_w^{-2})^{-1}n$ for large $n$.

Interestingly, eq. (10) has an exact solution, when the external noise is drawn from the Cauchy distribution, i.e., $f(ik) = \exp(-a|k|)$, In this case, it can be proved that $Z_\epsilon(k) = Z(ik) = \exp(-b|k|)$ with $b = a/(1 - r_w)$, which implies the Cauchy law

$$P(v) = \frac{1}{\pi b[1 + (v/b)^2]}.$$  \hspace{1cm} (19)

Note that this result is independent of the rate parameter $\gamma$ and the coefficient of restitution of the inter-particle collisions. It only depends on the coefficient of restitution $r_w$ for the wall-particle collision and the noise parameter $a$.

In conclusion, in this letter we have shown that it is possible to write exact recursion relation for the time evolution of the second moment and velocity correlation together for the driven inelastic Maxwell model. This enables us to obtain the form of the high-energy tails and all moments (recursively) of the velocity distribution in the steady state in the thermodynamic limit. We emphasize that we do not require any approximations to break the BBGKY hierarchy, as has been assumed in earlier work [20] but is unnecessary—the equations close once we also consider correlations [22]. Our results are also valid in the continuum time dynamics.

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