Fluctuation theorem for entropy production of a partial system in the weak-coupling limit

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Abstract – Small systems in contact with a heat bath evolve by stochastic dynamics. Here we show that, when one such small system is weakly coupled to another one, it is possible to infer the presence of such weak coupling by observing the violation of the steady-state fluctuation theorem for the partial entropy production of the observed system. We give a general mechanism due to which the violation of the fluctuation theorem can be significant, even for weak coupling. We analytically demonstrate on a realistic model system that this mechanism can be realized by applying an external random force to the system. In other words, we find a new fluctuation theorem for the entropy production of a partial system, in the limit of weak coupling.

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Introduction. – In systems where a few slow degrees of freedom (e.g., those of a colloidal particle in water) interact with a large number of fast degrees of freedom (e.g., those of the water molecules) —and there is a clear separation of timescales between the fast and slow degrees of freedom—the effects of the fast degrees of freedom on the slow degrees of freedom can be replaced by an effective white noise (and dissipation). This leads to a stochastic dynamics for the slow degrees of freedom where the fast degrees of freedom act as a heat bath. Within the framework of stochastic thermodynamics, the heat exchange between a stochastic system (of slow degrees of freedom) and a bath, and the work done on a stochastic system can be defined along individual stochastic trajectories [1–7]. While fluctuations of thermodynamic variables usually do not play any role in macroscopic systems, they are important for small systems [8–19] consisting of a few slow degrees of freedom, where the energies are comparable to $k_B T$. Over the past two decades or so, a number of remarkable mathematical relations have been found concerning the fluctuations of entropy [20–24], work [25,26], and heat [27–31].

The stochastic entropy production in a bath (medium) with a temperature $T$, due to an amount of heat $Q$ extracted from it by a stochastic system, is given by $\Delta S_{\text{med}} = -Q/T$. By assigning a certain entropy $S_{\text{sys}}$ for the stochastic system along the trajectories [24], the total entropy production is defined as $\Delta S_{\text{tot}} = \Delta S_{\text{med}} + \Delta S_{\text{sys}}$. In equilibrium $\Delta S_{\text{tot}} = 0$, whereas non-equilibrium processes generate entropy. The fluctuation theorem (FT) relates the probability density functions (PDFs) of positive and negative entropy productions in a given duration $t$, in the steady state of a non-equilibrium process, by [20–24]

$$\ln \left[ P_t(\Delta S_{\text{tot}})/P_t(-\Delta S_{\text{tot}}) \right] = \Delta S_{\text{tot}},$$

where Boltzmann’s constant is set to unity ($k_B = 1$).

Now consider a non-equilibrium system A coupled to another stochastic system B, with a dimensionless coupling strength $\delta$ (see fig. 1). The total entropy production of the combined systems A and B would evidently satisfy the FT, given by eq. (1). On the other hand, the FT is not expected to hold, if the total entropy production is measured by considering only system A [32–40]. Therefore, any deviations from the FT as given by eq. (1) would infer coupling to other stochastic processes. Clearly, without any coupling between A and B, the total entropy production
production for system A would again satisfy the FT. The question thus naturally arises whether there is any deviation from the FT, given by eq. (1), for vanishingly small coupling \( \delta \rightarrow 0 \).

In this letter, we show that under certain driving protocols, there can be significant deviations from eq. (1), even in the limit \( \delta \rightarrow 0 \), yielding a new FT for the entropy production of a partial system. This finding is in contrast with earlier studies [32–40], where the order of violation of the FT scales with the coupling strength which smoothly disappears in the limit of coupling strength going to zero. We propose a general mechanism, by which the FT for the partial entropy production can be broken under weak coupling. We demonstrate this mechanism for an experimentally realizable prototypical system.

**General mechanism.** – For a Markov process, to obtain the PDF \( P_t(\Delta S) \) of the entropy production \( \Delta S \) (or heat, work, etc.) in the steady state, in a given time \( t \), one usually needs to consider the joint PDF \( P_t(\Delta S, U) \) of \( \Delta S \) and all the relevant stochastic (slow) variables (denoted by the set \( U \)) that describe the system. The joint PDF satisfies a Fokker-Planck equation (FPE), \( \delta [\partial_t - \mathcal{L}] P_t(\Delta S, U) = 0 \), with the initial condition \( P_0(\Delta S, U) = \delta(\Delta S)\delta(U - U_0) \). Here, \( \tau \) is the characteristic time of the system. The Fokker-Planck operator \( \mathcal{L} \) involves differential operators with respect to \( U \) as well as \( \Delta S \) and the exact form of \( \mathcal{L} \) depends on the Langevin equations that describe the system. Integrating out \( \Delta S \) from the joint PDF yields the PDF of the stochastic variables \( P_t(U) = \int P_t(\Delta S, U) d\Delta S \) and in the limit \( t \rightarrow \infty \) we get the steady state PDF \( P_{t \rightarrow \infty}(U) \rightarrow P_{ss}(U) \). The PDF of the entropy production \( P_t(\Delta S) \) can be obtained from joint distribution \( P_t(\Delta S, U) \) by integrating out \( U \), and averaging over the initial variables \( U_0 \) with respect to the steady-state PDF \( P_{ss}(U_0) \). It turns out that the FT for the total entropy production \( \Delta S_{tot} \) of a complete stochastic system, as given by eq. (1), can be proven [24] without the explicit form of \( P_t(\Delta S_{tot}) \), and hence, the proof does not require solving the FPE. However, this is not the case for other quantities such as heat, work, or the entropy production of a partial system—which is the observable of interest of this letter. For these quantities, there is no general proof for a FT and in some cases, the FT (for those quantities) is not even satisfied. Therefore, for work, heat, entropy production of partial system, etc., one has to rely on the explicit form of the PDFs to make any statement. Unfortunately, in practice, finding the solution of the FPE is a non-trivial task and there exists only a few examples where the complete solution of the FPE can be obtained. Therefore, the next best thing is to try to obtain the PDFs for large \( t \), and examine whether the FT is satisfied at least for large \( t \). This is our goal in this letter.

Now let \( U_A \subset U \) be the set of stochastic variables that describe system A. The PDF \( P_t^A(U_A) \) at any time \( t \) can be found by keeping only the degrees of freedom of sub-system A and integrating out the rest from \( P_t(U) \). The system entropy for A can be defined as \[ S^A_{sys}(t) = -\ln P_t^A(U_A(t)) \]. Let \( \Delta S_{med}^A = -Q_A/T \) be the entropy production in the medium due to A, where \( Q_A \) is the heat transfer from the medium to A in a given duration. The joint PDF \( P_t(\Delta S_{med}^A, U) \) satisfies a FPE as mentioned above.

It is convenient to consider the generating function \( Z(\lambda, U, t|U_0) = \langle \exp(-\lambda \Delta S_{med}^A) \rangle_{(U|U_0)} \), where the expectation is taken over all trajectories of the system that evolve from a given initial configuration \( U_0 \) to a given final configuration \( U \) in a given duration \( t \). Clearly, \( Z(0, U, t|U_0) = P_t(U) \) with the initial condition \( P_0(U) = \delta(U - U_0) \). The FPE for \( P_t(\Delta S_{med}^A, U) \) would lead to a Fokker-Planck-like equation \( \partial_t - \mathcal{L}_\lambda Z(\lambda, U, t|U_0) = 0 \) with the initial condition \( Z(\lambda, U, 0|U_0) = \delta(U - U_0) \). The differential operator \( \mathcal{L}_\lambda \) reduces to \( \mathcal{L} \) for \( \lambda = 0 \). The solution for \( Z(\lambda, U, t|U_0) \) can be expressed in the eigenbasis of the operator \( \mathcal{L}_\lambda \) as

\[
Z(\lambda, U, t|U_0) = \sum_n \chi_n(U_0(\lambda))\Psi_n(U, \lambda)e^{(t/\tau)\mu_n(\lambda)}.
\]

Here \( \{\mu_n(\lambda)\} \) are the eigenvalues of \( \mathcal{L}_\lambda \) and \( \{\chi_n(U, \lambda)\} \) and \( \{\Psi_n(U, \lambda)\} \) are the left and right eigenfunctions, which satisfy the eigenvalue equation \( \mathcal{L}_\lambda\Psi_n(U, \lambda) = \mu_n(\lambda)\Psi_n(U, \lambda) \) and the orthonormality \( \int \chi_m(U, \lambda)\Psi_n(U, \lambda)dU = \delta_{m,n} \). The large time behavior is determined by the term containing the largest eigenvalue. Let \( \lambda(\mu) := \max\{\mu_n(\lambda)\} \) be the largest eigenvalue and \( \chi(\lambda, \lambda) \) and \( \Psi(\lambda, \lambda) \), respectively, be the corresponding eigenfunctions. Thus, for large time,

\[
Z(\lambda, U, t|U_0) = \chi(U(0, \lambda))\Psi(U, \lambda)e^{(t/\tau)\mu(\lambda)} + \cdots.
\]

Evidently, \( \mu(0) = 0 \), \( \chi(U_0, 0) = 1 \), and \( \Psi(U) = P_{t \rightarrow \infty}(U) \) is the steady-state PDF of \( U \). Consequently, the steady-state PDF \( P_{ss}^A(U_A) = P_{t \rightarrow \infty}(U_A) \) can be obtained from \( \Psi(U) \).

In the steady state, the change in the system entropy of A in duration \( t \), is given by \( \Delta S^A_{sys} = \ln[P^A_{ss}(U_A0)/P^A_{ss}(U_A)] \). Therefore, the generating function of the total entropy production, \( \Delta S^A_{tot} = \Delta S^A_{med} + \Delta S^A_{sys} \) of A, in the steady state can be obtained, using eq. (2) in \( Z(\lambda, U, t|U_0) \) \( \exp(-\lambda\Delta S^A_{sys}) \) and averaging over the initial condition \( U_0 \) with respect to the steady-state PDF \( \Psi(U_0) \) and integrating over the final variables \( U \), as

\[
\langle \exp(-\lambda\Delta S^A_{tot}) \rangle = g(\lambda)e^{(t/\tau)\mu(\lambda)} + \cdots,
\]

where

\[
g(\lambda) = \int dU_0 \Psi(U_0, 0) [P^A_{ss}(U_A0)]^{-\lambda} \chi(U_0, \lambda) \times \int dU[P^A_{ss}(U_A)]^\lambda \Psi(U, \lambda).
\]

The PDF is related to the above generating function, by the inverse transformation

\[
P_t^A(U_A) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \langle \exp(-\lambda\Delta S^A_{tot}) \rangle e^{\lambda\Delta S^A_{tot}} d\lambda.
\]
Therefore, for large $t$, the PDF of the time-averaged total entropy production $s \equiv (t/\tau)^{-1} \Delta S^\lambda_{\text{tot}}$ is given by
\[
p(s) = \frac{(t/\tau)}{2\pi i} \int_{1-i\infty}^{1+i\infty} g(\lambda) e^{(t/\tau)[\mu(\lambda)+\lambda s]} d\lambda + \cdots. \quad (4)
\]

First consider the case in which system A is isolated from other stochastic systems ($\delta = 0$). Here we use the notations $g_0(\lambda)$ and $\mu_0(\lambda)$ in place of $g(\lambda)$ and $\mu(\lambda)$, respectively. For this isolated system, the FT as in eq. (1) must hold for $\Delta S^\lambda_{\text{tot}}$, that is, $p(s)/p(-s) = \exp[(t/\tau)s]$. From eq. (4), with the change of the integration variable $\lambda \to 1 - \lambda$, we get
\[
e^{(t/\tau)s} p(-s) = \frac{(t/\tau)}{2\pi i} \int_{1-i\infty}^{1+i\infty} g_0(1-\lambda) e^{(t/\tau)[\mu_0(1-\lambda)+\lambda s]} d\lambda + \cdots.
\]

Note that the contour of integration $C$ in the above integral is parallel to the imaginary axis through $\text{real}(\lambda) = 1$. Now, for the right-hand side to be equal to $p(s)$, we require the Gallavotti-Cohen (GC) symmetry $\mu_0(\lambda) = \mu_0(1-\lambda)$ and $g_0(\lambda) = g_0(1-\lambda)$ to hold, and both $\mu_0(\lambda)$ and $g_0(\lambda)$ to be analytic, at least within the region between the imaginary axis and the contour $C$, so that the contour $C$ can be shifted to the imaginary axis through the origin without any additional contribution from singularities. For $t \gg \tau$, the saddle-point approximation of eq. (4) gives
\[
p(s) = \sqrt{\frac{(t/\tau)}{2\pi i\mu_0'(g_0)} g_0(\lambda_0^\mu) e^{(t/\tau)[\mu_0(\lambda_0^\mu)+\lambda_0^\mu s]} + O \left( \frac{\tau}{\sqrt{\tau}} \right)},
\]
where the saddle point $\lambda_0^\mu(s)$ is given by
\[
\mu_0'(\lambda_0^\mu) = -s. \quad (5)
\]

The saddle point satisfies the symmetry $\lambda_0^\mu(s) + \lambda_0^\mu(-s) = 1$. Now, ignoring the subleading prefactor, one gets the so-called large deviation form [41] $p(s) \sim \exp[(t/\tau)I_0(s)]$, where the function $I_0(s)$ is usually called the large deviation function (LDF), given by $I_0(s) = \mu_0(\lambda_0^\mu) + \lambda_0^\mu s$. The GC symmetry implies the symmetry
\[
I_0(s) - I_0(-s) = s, \quad (6)
\]
which is equivalent to the FT as in eq. (1) for large $t$. In this letter, our aim is to investigate, whether such a relation is valid for non-zero $\delta$, in the limit $\delta \to 0$.

Let us consider the situation where $\mu_0(\lambda)$ is analytic only within a finite region bounded by a pair of branch point singularities at $\lambda_\pm$. For the FT to hold, $g_0(\lambda)$ must be analytic within this region $\lambda \in (\lambda_, \lambda_+)$, with $\lambda_0 < 0$ and $\lambda_0 + 1$ with $\lambda_+ + \lambda_- = 1$. Moreover, $\mu_0(\lambda_+) = \mu_0(1-\lambda_-) = \mu_0(\lambda_-)$. We assume that near these branch points \footnote{This specific branch point behavior is taken only as an explicit example. It is not necessary to have this specific form and one can in fact have other branch point behaviors such as the logarithmic one. The nature of the branch point singularity is not important as it only contributes to the subleading correction.}
\[
\mu_0(\lambda) = \begin{cases} 
\mu_0(\lambda_+) - b(\lambda_+ - \lambda)^{\rho_0} + \cdots & \text{as } \lambda \to \lambda_+, \\
\mu_0(\lambda_-) - b(\lambda - \lambda_-)^{\rho_0} + \cdots & \text{as } \lambda \to \lambda_-,
\end{cases}
\]
where $0 < \rho_0 < 1$, and $b$ is a constant. From the saddle-point equation (5), it follows that $\lambda_0^\mu(s) \to \lambda_\pm$ at the leading order as $s \to \pm\infty$. Consequently $I_0(s) = \mu_0(\lambda_\pm) + \lambda_\pm s + \cdots$ at the leading order in $s$, as $s \to \pm\infty$. Since eq. (6) is valid for all $s$, the subleading correction terms to the relation $I_0(s) - I_0(-s) = [\lambda_+ + \lambda_-]s$ vanish at all orders.

In the presence of a non-zero coupling ($\delta > 0$), let us suppose that $\mu(\lambda)$ has branch points at $\lambda_\pm^{(\delta)}$. The saddle-point approximation of eq. (4) gives the large deviation form $p(s) \sim \exp[(t/\tau)I(s)]$. If $g(\lambda)$ is analytic in the region $\lambda \in (\lambda_-, \lambda_+^{(\delta)})$, then the LDF is given by $I(s) = \mu(\lambda^*) + \lambda^* s$ with $\mu(\lambda^*) = -s$, as in the $\delta = 0$ case. In case $g(\lambda)$ has a singularity within this range, it can change the LDF [42–44]. However, eventually, we are interested in the $\delta \to 0$ limit, where we can write $g(\lambda) = g_0(\lambda) + \delta g_1(\lambda) + \cdots$, with $\delta > 0$, and the function $g_1(\lambda)$ may have singularities. Therefore, the integral in eq. (4) can be written as the sum of two integrals, one with a prefactor proportional to $g_0(\lambda)$ and the other with the prefactor proportional to $\delta g_1(\lambda)$. It is evident that the second integral would not contribute in the $\delta \to 0$ limit. Therefore, we only consider the integral with the prefactor proportional to $g_0(\lambda)$. As in the $\delta = 0$ case, here we get $I(s) = \mu(\lambda_\pm^{(\delta)}) + \lambda_\pm^{(\delta)} s + \cdots$ as $s \to \pm\infty$, at the leading order in $s$, where $\lambda_\pm^{(\delta)}$ represents the analytical part of $\mu(\lambda)$ at the respective branch points. This implies that the asymmetry function,
\[
f(s) = I(s) - I(-s), \quad (7)
\]
has the asymptotic form
\[
f(s) = [\mu(\lambda_\pm^{(\delta)}) - \mu(\lambda_\pm)] + [\lambda_\pm^{(\delta)} + \lambda_\pm^{(\delta)}] s + \cdots \text{ as } s \to \infty.
\]
Since, we do not expect $\mu(\lambda)$ to obey the GC symmetry, the slope $[\lambda_\pm^{(\delta)} + \lambda_\pm^{(\delta)}]$ need not be unity and $f(s)$ can have subleading corrections in $s$. Therefore, for a finite $\delta$, the deviations from the straight line $f(s) = s$, provides a measure of the violation of the FT. Now the question is whether such deviation persists in the limit $\delta \to 0$.

To answer this question, we note that, purely on the general ground, in the limit $\delta \to 0$, there can be four distinct possibilities:

- (P1) $\lambda_\pm^{(\delta)} \to \lambda_\pm$,
- (P2) $\lambda_\pm^{(\delta)} \to \lambda_+$ and $\lambda_-^{(\delta)} \to \tilde{\lambda}_-$,
- (P3) $\lambda_-^{(\delta)} \to \lambda_-$ and $\lambda_+^{(\delta)} \to \tilde{\lambda}_+$,
- (P4) $\lambda_\pm^{(\delta)} \to \tilde{\lambda}_\pm$.

Note that the original contour of integration is along the imaginary axis through the origin. Therefore, the two singularities, one on each side of the origin, that are closest to the origin from the respective side only matter, as the saddle point is bounded by these two closest singularities. Hence, $\lambda_- < \tilde{\lambda}_- < 0$ and $0 < \tilde{\lambda}_+ < \lambda_+$. 

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Clearly, the FT is not violated for the case (P1) in eq. (8) and one gets the same LDF $I_0(s)$ obtained above for the uncoupled case. Now, consider the situation (P2). In this case, near $\lambda_-$, in the limit $\delta \to 0$ we can write (see footnote 1): $\mu(\lambda) = \mu_0(\lambda) - a\delta(\lambda - \lambda_+)^{\rho}$, where $\beta > 0$, $0 < \rho < 1$, and $a$ is a constant. The saddle-point equation $\mu^\prime(\lambda^*) = -s$, yields

$$\mu_0^\prime(\lambda^*) = \frac{a\rho\delta}{(\lambda^* - \lambda_-)^{1-\rho}} = -s. \quad (9)$$

We have found above for the $\delta = 0$ case that the saddle point given by eq. (5) stays between $(\lambda_-, \lambda_+)$. Therefore, for $\delta \to 0$, if $\lambda^*(\delta) \to \lambda_-\pm$ instead of $\lambda_-$, then it is necessary that $\lambda_- < \lambda_- < 0$. In the limit $\delta \to 0$, when $s$ increases from $-\infty$ to $\infty$, the saddle point $\lambda^*(s)$ moves from $\lambda_+ \to \lambda_-\pm$ on the real line $\lambda$. It is evident from eq. (9) that for $[\lambda^*(s) - \lambda_-] \gg \delta^{3/(1-\rho)}$, the left-hand side of eq. (9) is dominated by the first term and therefore the saddle point is given by the equation $\mu_0^\prime(\lambda^*) = -s$. Consequently, the LDF is the same $I_0(s)$, that has been obtained for the uncoupled case. On the other hand, we see that $\lambda^*(s) \to \lambda_-\pm$ for small $s$, whereas the first derivative $\mu^\prime(\lambda^*)$ changes sign at $s = 0$. Thus, in the limit $\delta \to 0$, we get

$$I(s) = \begin{cases} I_0(s), & \text{for } s < s_1^*, \\ \mu_0(\lambda_-) + \lambda_- s, & \text{for } s > s_1^*, \end{cases}$$

where $s_1^*$ is given by $\lambda_0^*(s_1^*) = \lambda_-\pm$. Similarly for (P3) in eq. (8), we get

$$I(s) = \begin{cases} \mu_0(\lambda_+) + \lambda_+ s, & \text{for } s < s_2^*, \\ I_0(s), & \text{for } s > s_2^*, \end{cases}$$

where $s_2^*$ is given by $\lambda_0^*(s_2^*) = \lambda_+\pm$. Here $s_2^* < s_1^*$, as $\lambda_+ > \lambda_-\pm$. Finally, for (P4), in the limit $\delta \to 0$, we get

$$I(s) = \begin{cases} \mu_0(\lambda_-) + \lambda_- s, & \text{for } s < s_3^*, \\ I_0(s), & \text{for } s_2^* < s < s_1^*, \\ \mu_0(\lambda_-) + \lambda_- s, & \text{for } s > s_1^*. \end{cases}$$

We find that usually $f(s) = s$ for small $s$, except for $s_2^* > 0$ where one has $f(s) = 2\lambda_- s$ for small $s$. For large $s$, for the possibilities (P2)–(P4), $f(s)$ differs significantly from the small-$s$ behavior. From the second-order discontinuities of $I(s)$ at the points $s_1^*\pm$, it follows that $f(s)$ also exhibits second-order discontinuities at these points. The asymptotic expression of $f(s)$, as $s \to \infty$, is given by

$$f(s) = \begin{cases} s, & \text{for (P1),} \\ [\mu_0(\lambda_-) - \mu_0(\lambda_+)] + [\lambda_- + \lambda_+] s, & \text{for (P2),} \\ [\mu_0(\lambda_-) - \mu_0(\lambda_+)] + [\lambda_- + \lambda_-] s, & \text{for (P3),} \\ [\mu_0(\lambda_-) - \mu_0(\lambda_+)] + [\lambda_+ + \lambda_-] s, & \text{for (P4)} \end{cases}$$

and $f(-s) = -f(s)$, where (P1)–(P4) represent the four cases given in eq. (8). Thus, if the analytic region of $\mu(\lambda)$ is bounded by a pair of branch points, such that in the limit $\delta \to 0$, at least one of the limiting branch points differs from that of the uncoupled case ($\delta = 0$), then for large $s$, the slope of the asymmetry function differs from unity. This prominent deviation is indeed an indication of coupling to an external system. Equation (13) is our main result, which provides new FT for the entropy production of a partial system in the weak-coupling limit. In the following, we demonstrate that the above mechanism can indeed be realized in real systems by subjecting it to an external stochastic forcing.

The model (coupled Brownian motion). – We consider a system of two Brownian particles (denoted by A and B) coupled to each other by a harmonic potential $U(y) = ky^2/2$, where $y$ is the separation between them and $k$ is the spring constant. For simplicity, we set the masses of both the particles to be equal to $m$. The Hamiltonian of the coupled system is thus given by

$$H = \frac{\hbar}{2}/(v_A^2 + v_B^2) + U(y),$$

where $v_A$ and $v_B$ are the velocities of the particles A and B, respectively. The whole system is in contact with a heat bath at a temperature $T$. We apply an external Gaussian stochastic force $f_A(t)$, with mean zero and correlator $\langle f_A(t) f_A(t') \rangle = 2\beta^2 \delta(t - t')$, on the particle A. It is possible that the particle B also experiences an external force $f_B(t)$ (which is assumed to be Gaussian with mean zero), that may be either correlated or independent to the force applied on the particle A. Therefore, consider two opposite situations: i) where $f_B(t)$ is independent of $f_A(t)$ (with $\langle f_B(t) f_B(t') \rangle = 2\alpha \beta^2 \delta(t - t')$ and $\alpha = \beta^2 / (\gamma T)$, where $\gamma$ is the friction coefficient. The dynamics of the system is described by the coupled Langevin equations,

$$\dot{y} = v_A - v_B,$$

$$\dot{v}_A = - \gamma v_A(t) - ky_A(t) + \eta_A(t) + f_A(t),$$

$$\dot{v}_B = - \gamma v_B(t) + ky_B(t) + \eta_B(t) + f_B(t),$$

where $\gamma = \beta^2 / (\gamma T)$.
where \( \eta_A(t) \) and \( \eta_B(t) \) are Gaussian white noise due to the thermal bath acting on the particles A and B, respectively. The mean \( \langle \eta_A(t) \rangle = \langle \eta_B(t) \rangle = 0 \) and the the correlations \( \langle \eta_A(t) \eta_B(t') \rangle = 2\gamma \delta(t - t') \), whereas \( \eta_A \) and \( \eta_B \) are independent of each other as well as independent of the external stochastic Gaussian forces \( f_A \) and \( f_B \).

Partial entropy production (definition). – The heat transfer from the bath to particle A in a time duration \( t \) is given by [1], \( Q_A = \int_0^t \langle \dot{\eta}_A(t') - \gamma v_A(t') \rangle v_A(t') \, dt' \), where \( \eta_A(t') \) is the Gaussian white noise acting on particle A due to the thermal bath. Consequently, \( \Delta S_A^{\text{med}} = -Q_A/T \) is the entropy production in the medium due to particle A. In the steady state, the change in the system entropy for particle A is given by [24], \( \Delta S_A^{\text{sys}} = \ln[P_A^\text{ss}(v_A(t))/P_A^\text{ew}(v_A(t))] \), where \( P_A^\text{ss}(v_A) \) is the steady-state distribution of the velocity \( v_A \). The PDF of the total entropy production, \( \Delta S_A^{\text{tot}} = \Delta S_A^{\text{med}} + \Delta S_A^{\text{sys}} \), for the partial system A, for large \( t \), satisfies eq. (4), with \( \tau = \gamma t = m/\gamma \) being the viscous relaxation time.

Apparent entropy production (definition). – While \( \Delta S_A^{\text{med}} \) considered above is the true partial entropy production in the medium due to the system A, an experimental observer without any knowledge about the coupling with system B would model system A with \( k = 0 \). This gives the entropy production of the medium due to A in terms of the experimentally obtainable stochastic trajectories as \( \Delta S_A^{\text{med}} = W - \frac{m}{\gamma} [\langle \dot{v}_A(t) \rangle - \langle \dot{v}_A(0) \rangle] \), where \( W = \frac{1}{2} \int_0^t f_A(t') v_A(t') \, dt' \). Similarly, the change in the system entropy would be defined as \( \Delta S_A^{\text{sys}} = \ln[P_A^\text{sys}(v_A(t))/P_A^\text{ew}(v_A(t))] \), where \( P_A^\text{sys}(v_A) \) is the steady-state distribution of the velocity \( v_A \) obtained for \( k = 0 \). By adding both parts, we call \( \Delta S_A^{\text{app}} = \Delta S_A^{\text{med}} + \Delta S_A^{\text{sys}} \) as apparent entropy production due to system A. However, to compute this apparent entropy production, we use the Langevin equations for the full system A and B with \( k \neq 0 \), as in reality, there is a non-zero coupling. Evidently, the two definitions of the entropy coincide for \( \delta = 0 \).

Methods for computation of \( \mu(\lambda) \) and \( g(\lambda) \). – It is convenient to use the Fourier transforms in the time domain \((0, t)\). Since eqs. (14) are linear, the Fourier transforms \((\tilde{g}, \tilde{\dot{v}}_A, \tilde{\dot{v}}_B)\), depend linearly on \((f_A, f_B, \tilde{\eta}_A, \tilde{\eta}_B)\). Consequently, \( \Delta S_A^{\text{med}} \) and \( W \) are quadratic in \((f_A, f_B, \tilde{\eta}_A, \tilde{\eta}_B)\). Since, stochastic forces and the thermal noises are uncorrelated in time, their Fourier transforms for any frequency \( \omega \) are correlated to only negative frequency \(-\omega\). Moreover, since \((f_A, f_B, \tilde{\eta}_A, \tilde{\eta}_B)\) are real variables, their Fourier transforms at a negative frequency \(-\omega\) are equal to the complex conjugate of the corresponding Fourier transforms with positive frequency \(\omega\), i.e., \( f_A(-\omega) = \tilde{f}_A(\omega) \) and so on. Therefore, using the Gaussian distribution of \((f_A, f_B, \tilde{\eta}_A, \tilde{\eta}_B)\) independently for each frequency, we can express the generating functions \( \exp(-\Delta S_A^{\text{med}}) \) and \( \exp(-\Delta S_A^{\text{sys}}) \) as infinite product of independent Gaussian integrals for each frequency. Finally, in the large-\( t \) limit, by considering the frequency to be continuous, and integrating over the final phase-space variables and averaging over the initial phase-space variables, we obtain [see 45] for details on a similar derivation] the generating function similar to eq. (3) with \( \tau = m/\gamma \). Since, system entropy productions depend only on the initial and final velocities, and not on the full trajectory, they do not contribute to \( \mu(\lambda) \), but contribute only to the prefactor \( g(\lambda) \). The calculations are carried out separately for the four cases: i.e., the two definitions of the entropy and the two choices of the stochastic force \( f_B \). In all the cases, \( \mu(\lambda) \) has the integral form

\[
\mu(\lambda) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} du \ln \left[ 1 + \frac{b(u, \lambda)}{q(u)} \right],
\]

where the functions \( b(u, \lambda) \) and \( q(u) \) are different for each case. The expressions of \( g(\lambda) \), for all the cases are quite involved and not very illuminating. Fortunately, we are interested in the limit of \( \delta \to 0 \) (i.e., \( k \to 0 \)), and for that, as discussed above, we only need \( g_0(\lambda) \). This is given by a simple expression (see footnote 2), \( g_0(\lambda) = 2\sqrt{\nu(\lambda)}/[1 + \nu(\lambda)] \) with \( \nu(\lambda) = \sqrt{1 + 4\lambda(1 - \lambda)} \).

Partial entropy production (results). – For the uncoupled case \( \delta = 0 \), we find that (see footnote 2) \( \mu_0(\lambda) = \frac{1}{2}[1 - \nu(\lambda)] \) has branch point singularities at \( \lambda_{\pm} = \frac{1}{2}[1 \pm \sqrt{1 + \theta^2}] \) and is analytic in the region bounded by these branch points. We also find that \( I_0(s) \)

2Details will be published elsewhere.
Fig. 3: (Color online) The asymmetry functions corresponding to different regions of the parameter space of fig. 2: (a) and (b) for regions (P3) and (P4), respectively in fig. 2(a). (c) and (d) for the regions (P3) and (P4), respectively in fig. 2(b). (e) for regions (P1) of fig. 2(c) as well as fig. 2(d). (f) and (g) for the regions (P3) and (P4), respectively, in fig. 2(c). (h) for region (P2) in fig. 2(d). In all the figures the green dashed lines through the origin plot the function $f(s) = s$ and the magenta dashed lines plot the asymptotic predictions of $f(s)$ for large $s$, given by eq. (13) — the points where these two lines meet do not have any significant. The orange dotted lines (marked by $\delta = 0$) plot the limiting expressions of $f(s)$, obtained from the expressions of $I(s)$ given in the text.

obeyes eq. (6). Now, in the presence of a non-zero coupling $\delta$, we find that $\lambda^{(\delta)}_+ = \lambda_+ = 1$ for all $0 < \delta < 1$, including the limit $\delta \to 0$. On the other hand, the limiting behavior of $\lambda^{(\delta)}_-$ depends on the parameters ($\alpha, \theta$) as shown in fig. 2, where the phase boundaries can be calculated exactly (see footnote 2). In the (P3) regions in figs. 2(a) and (b), we get $\lambda^{(\delta)}_+ \to \lambda_+$ as $\delta \to 0$, whereas in the (P4) regions in figs. 2(a) and (b), we get $\lambda^{(\delta)}_+ = \lambda_-$ for all $0 < \delta < 1$. Here $\lambda_+ = -[1 + (1 + \alpha^2 \theta)^{-1}]$ for the choice i) of $f_B(t)$ and $\lambda_- = -[1 + (1 + \alpha^2 \theta)^{-1}]$ for the choice ii). Therefore, in the limit $\delta \to 0$, while $f(s) \approx s$ for small $s$, it is given by eq. (13) for large $s$.

Apparent entropy production (results). – In the presence of a non-zero $\delta$, for the choice i) of $f_B(t)$, we find that the possibilities (P1), (P3) and (P4) can be realized (fig. 2(c)) depending on the values of the driving parameters (see footnote 2), and $\lambda_+$ given by $\lambda_+ = \{(\theta + \sqrt{\theta}[2 + (1 + \alpha^2 \theta)]) / (\theta[2 + \alpha^2 \theta])\}$. On the other hand, for the choice ii) of $f_B(t)$, the possibilities (P1) and (P2) can be realized (fig. 2(d)) and $\lambda_+ = \{(1 + \alpha)(\theta - \sqrt{\theta}[2 + (1 + \alpha^2 \theta)]) / (2\theta)\}$.

Numerical comparison. – In fig. 3 (also see footnote 2), we compare the theoretical prediction of $f(s)$ for large $s$ in the limit $\delta \to 0$, given by eq. (13), against exact results obtained by numerically inverting $\mu(\lambda)$ for small $\delta$, for both the choices of the driving force $f_B(t)$ and also for both the considerations of the entropy production. We find that, as $\delta$ decreases, the numerical curves converge to the limiting (limit $\delta \to 0$) expressions of $f(s)$ (see footnote 2). Moreover, for large $s$, they converge to the asymptotic expressions given in eq. (13). We also refer to footnote 2 for comparisons of the PDF and LDF with numerics.

System in a trap. – Finally, we ask whether the effect of the external weak coupling can be nullified. Indeed, we find that (see footnote 2), when the system is placed in a harmonic trap, we always get $\lambda^{(\delta)}_+ = \lambda_+$ as $\delta \to 0$. Thus, in this case, the relation $f(s) = s$ is always satisfied in the limit $\delta \to 0$. This suggests that weak coupling cannot affect the FT in the presence of a trap, and hence, this provides a way to neutralize the influence of such coupling.

Concluding remarks. – We have found a new mechanism by which the FT of the entropy production can be violated in the presence of a coupling to an external system, even in the limit of the coupling strength going to zero. In other words, we have provided a new FT for the entropy production of a partial system, in the presence of weak coupling, driven by external random forces. Conversely, our finding gives a way to find out if a particular stochastic process of interest is coupled to any other hidden stochastic systems. Thus, it provides a new application of FT that can be applied to a wide variety of small systems.

Our result may look quite surprising at first glance, as it goes against our naive intuition that the effect of coupling should disappear in the limit of interaction strength going to zero, as in the regular perturbation problems. However, it should be emphasized that, here, the limit of coupling
strength going to zero is a singular perturbation, which is very different from the case of coupling strength equal to zero. For example, in the case of the two coupled Brownian particles A and B (without the external forces), the coupling introduces a timescale $\tau_k = \gamma/k$, beyond which the separation between the particles relaxes to the equilibrium with $\langle \gamma \rangle_{eq} = DT\tau_k$, whereas for $\tau_k \ll \tau_k$ we have $\langle \Delta y^2 \rangle = 4Dt$, where $D$ is the diffusion constant. Therefore, while initially the particle A (or B) behaves like a free particle with $\langle \Delta x^2 \rangle = 2Dt$ for $\tau_k \ll \tau_k$, beyond the timescale $\tau_k$, it diffuses as the center of mass with $\langle \Delta x^2 \rangle = (D\tau_k)/4 + Dt$ for $t \gg \tau_k$. This example indirectly demonstrates that it is possible to observe the effect of weak coupling if one looks at a time beyond the timescale introduced by the coupling. Note that in this letter, we have already taken the large time limit before taking the $\delta \to 0$ limit. However, it does not directly explain our results, as we have found that in the regions (P1) of figs. 2(c) and (d), the FT is satisfied. A clear understanding of how different timescales can lead to the singular limit remains an open problem. It would be interesting to demonstrate the crossover from the validity of the FT at small time to the singular limit at long time, through a model that can be exactly solved for all time (see footnote 2).

The harmonic trap (of stiffness $k_0$) introduces a timescale $\tau_{k_0} = \gamma/k_0$ beyond which the system relaxes to the steady state. The weak coupling changes this timescale only by a small amount $\tau_k = \gamma/k_0[1 + O(k/k_0)]$. Perhaps, for this reason, in the presence of a trap, the FT is always satisfied in the limit of coupling strength going to zero.

REFERENCES