Bangalore School on Statistical Physics XIV, Sept 2023

Statistical Mechanics of Complex Networks

Lecture 3: Models (Random, Small-world & all that!)

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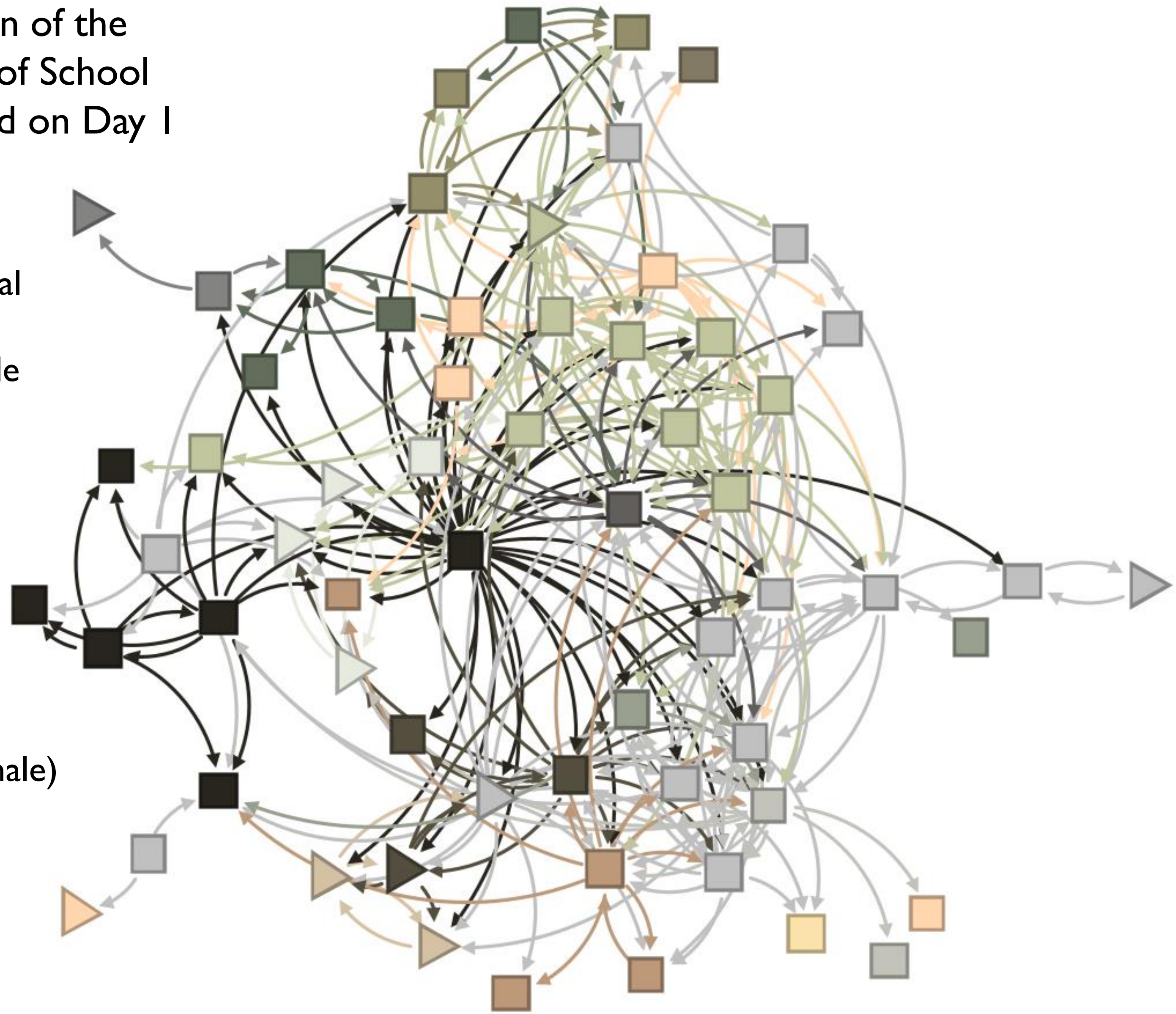
The Institute of Mathematical Sciences, Chennai

A partial reconstruction of the acquaintance network of School participants as reported on Day 1

Node color → institutional affiliations of participants (institutes with just a single participant colored grey)

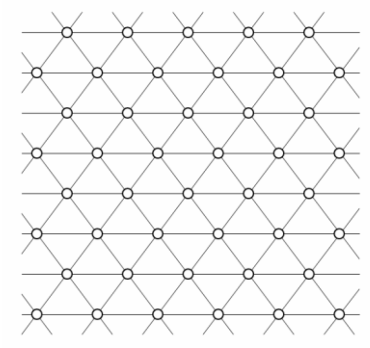
Node shape → gender (square: male, triangle: female)



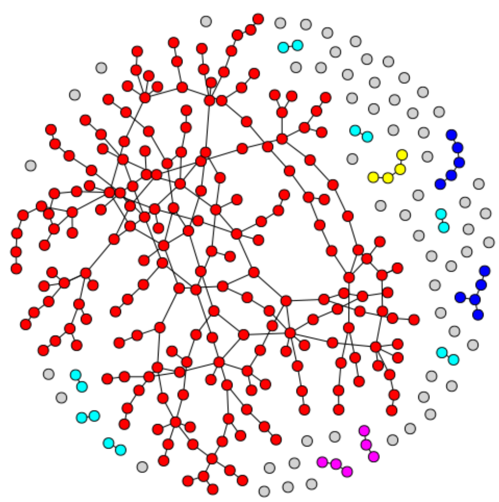
Data acquisition by Ritam Pal (IISER Pune)
Visual rendering by Shakti N Menon (IMSc Chennai)

Theoretical understanding of networks

Behavior of theoretical networks typically examined in the limit $N \to \infty$



- Regular lattice or grid (*Physics*)
  - average path length ~ $N^{1/D}$ (no. of nodes)
    D: spatial dimension
  - clustering *high*
  - delta function distribution of degree (links/node)

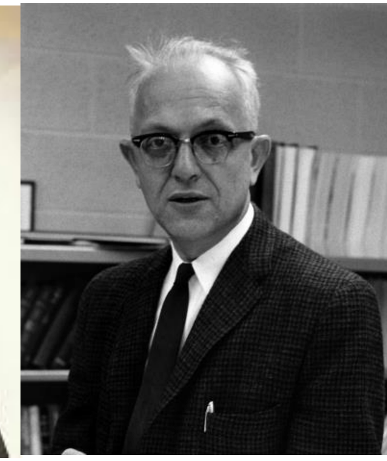


- Random networks (*Graph theory*)
  Also known as Erdös-Renyi networks
  - average path length ~ log N
  - clustering low
  - Poisson distribution of degree

Random networks

Networks constructed by choosing to place links between each possible pair of nodes using independent, identical probability

Ray Solomonoff



Anatol Rapoport

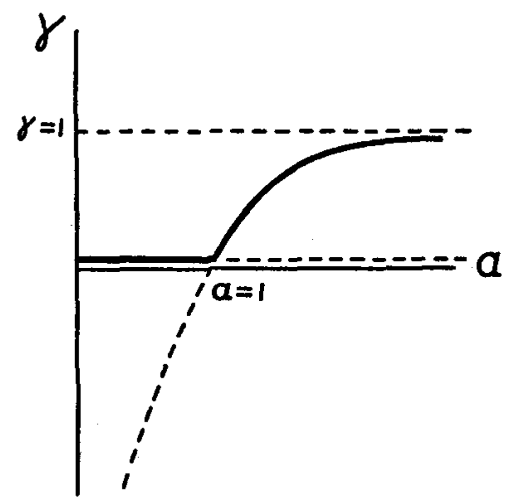
BULLETIN OF
MATHEMATICAL BIOPHYSICS
VOLUME 13, 1951

CONNECTIVITY OF RANDOM NETS

RAY SOLOMONOFF AND ANATOL RAPOPORT
DEPARTMENT OF PHYSICS AND COMMITTEE ON MATHEMATICAL BIOLOGY
THE UNIVERSITY OF CHICAGO

The weak connectivity γ of a random net is defined and computed by an approximation method as a function of a, the axone density. It is shown that γ rises rapidly with a, attaining 0.8 of its asymptotic value (unity) for $a = 2$, where the number of neurons in the net is arbitrarily large. The significance of this parameter is interpreted also in terms of the maximum expected spread of an epidemic under certain conditions.

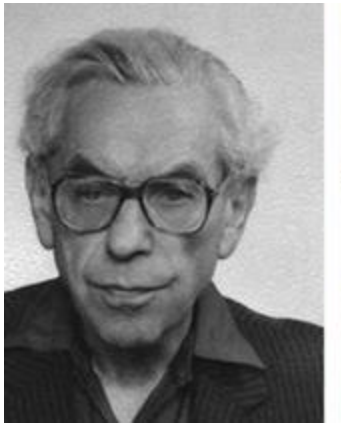
Numerous problems in various branches of mathematical biology lead to the consideration of certain structures which we shall call "random nets." Consider an aggregate of points, from each of which issues some number of outwardly directed lines (axones). Each axone terminates upon some point of the aggregate, and the probability that an axone from one point terminates on another point is the same for every pair of points in the aggregate. The resulting configuration constitutes a *random net*.



Weak connectivity as a function of axone density

Random networks

Erdös-Renyi model (1959): Two closely related probability-based models for generating random networks



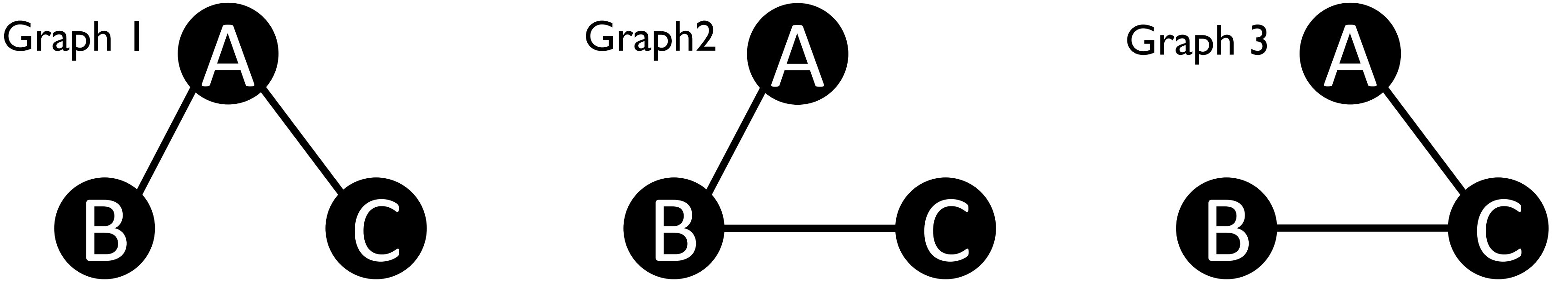
Paul Erdös

Alfred Renyi

I. The G(N,L) model: when any member of a family of all graphs with N nodes and L links is chosen uniformly at random.

Example:

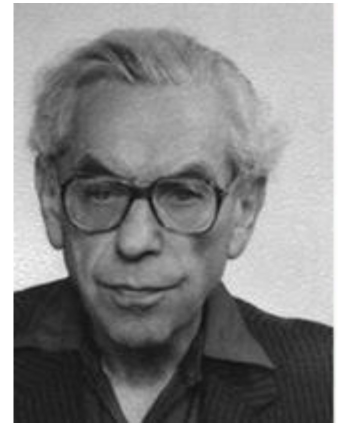
G(3,2) comprises 3 possible networks of 3 nodes A,B and C



Each graph can be picked with probability 1/3

Random networks

Erdös-Renyi model (1959): Two closely related probability-based models for generating random networks



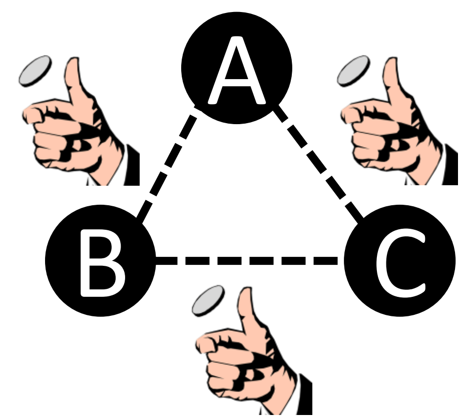
Paul Erdös

Alfred Renyi

II. The G(N,p) model: when a network is constructed by randomly placing a link between each possible pair of nodes with a probability p ($0<p<1$)

Example:

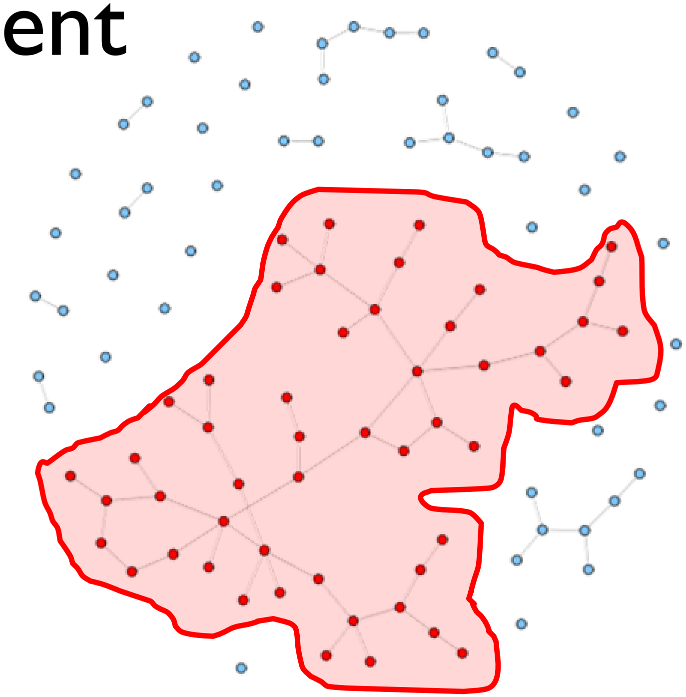
G(3, ½) is ensemble of all possible networks of 3 nodes A,B and C such that each link {AB, BC, AC} occurs with probability ½



As $N \to \infty$, $p \geq 2\ln(N)/N \Rightarrow$ network will almost surely be connected

Image: www.sci.unich.it

Largest Connected Component



A largest connected component (LCC, or giant component) is a connected component whose size N_l is a finite fraction of that of the size N of the entire network, even as the network becomes larger and larger, i.e., $\lim_{N\to\infty} N_l / N = c > 0$.

In G(N,p) random network model, LCC size is
$N_l = 1$ when p = 0 (isolated nodes)
$N_l = N$ when p = 1 (clique)

Phase transition

As p gradually increased from 0 to 1, the fraction N_l / N suddenly increases from 0 to a finite value (>0)
i.e., becomes extensive, increasing with N
at a critical value of p, $p_c = 1/N$

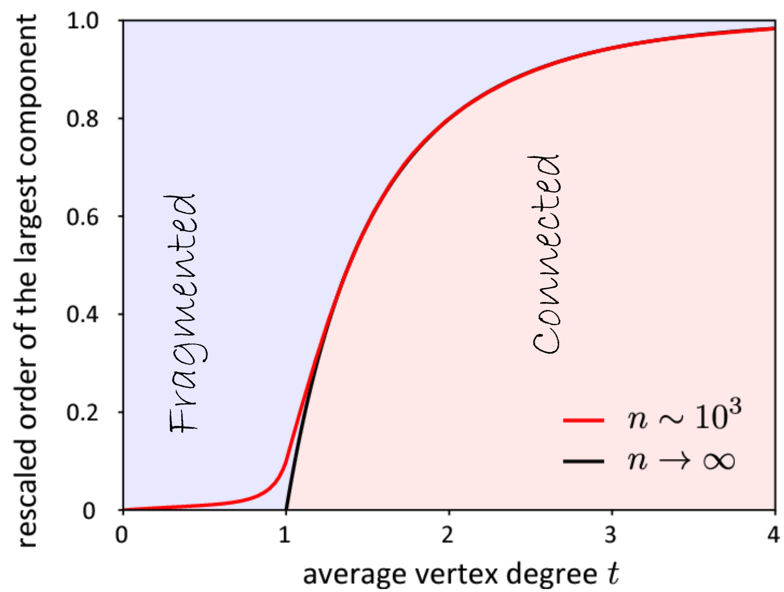
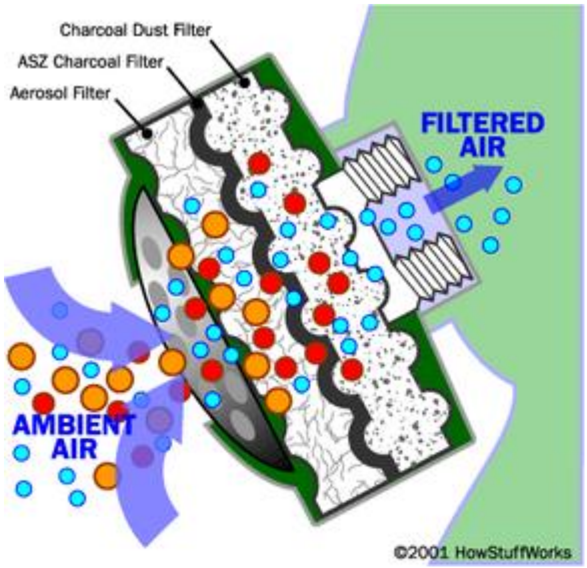


Image: Kang & Petrasek, Internat. Math. Nachrichten (2014)

Random networks & Percolation

Random Network phase transition related to bond percolation



Percolation theory has origins in understanding the process of transport through porous medium, e.g., of toxic chemical molecules through the filtering agent of a gas mask

Consider a 2-dimensional lattice of $N \times N$ sites in which the links between any two neighboring sites is open with probability p [⇒ closed with probability (1 – p)]

The Question:

What is the probability that a connected path exists from one side of the lattice to the other ?

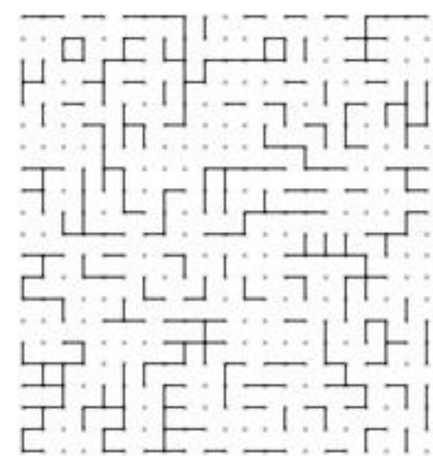


Image: http://www.suffos.bam.de

Path length & Clustering in Random networks

The average path length in the random network is $\langle \mathcal{L} \rangle \sim \log \langle N \rangle / \log \langle k \rangle$

Intuition:
Locally, a random network G(N,p) with very small *p* – as cycles or closed loops involving only a few nodes are unlikely – will be approximately like a tree

The average number of neighbors located at distance d away from a node is :
$N_d = \langle k \rangle^d$
$\Rightarrow N = \langle k \rangle + \langle k \rangle^2 + \langle k \rangle^3 + \ldots + \langle k \rangle^d \sim \langle k \rangle^d$

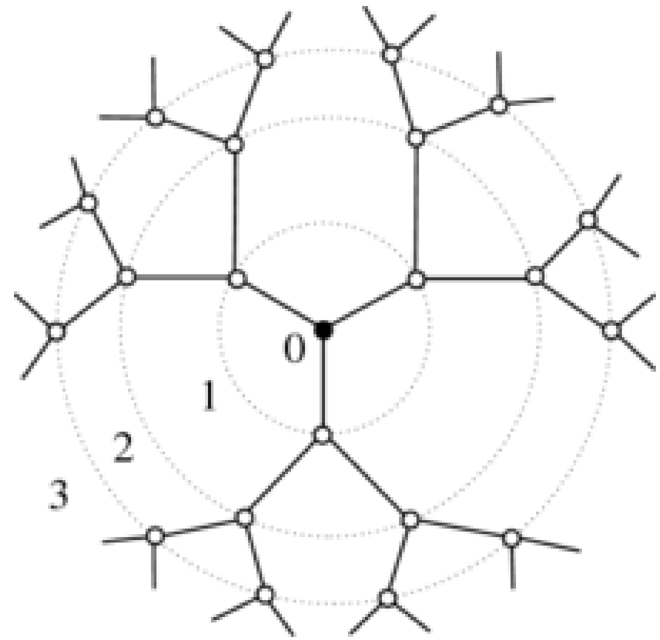


Image: http://i.stack.imgur.com

Bethe Lattice

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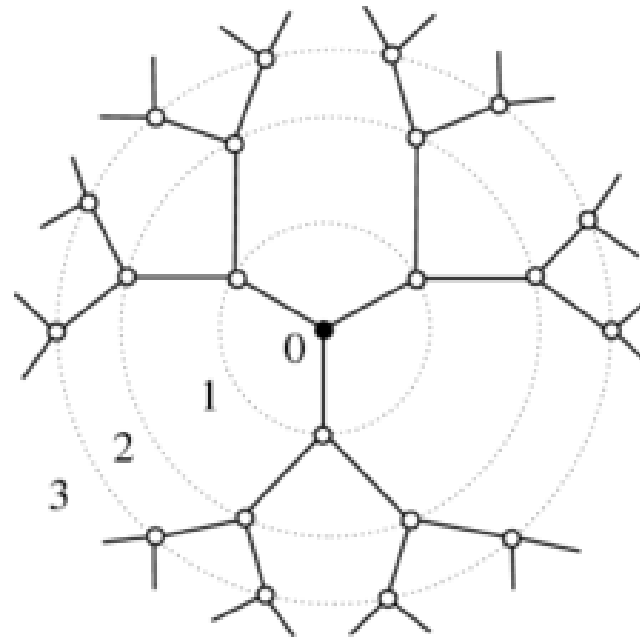

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The average clustering coefficient in a G(N,p) random network is approximately $\langle C \rangle \sim p \approx \langle k \rangle / N$
This is because if you randomly select a node i and look at two neighboring nodes j and k connected to it, the probability that j & k will be connected is just *p*

Degree distribution of Random networks

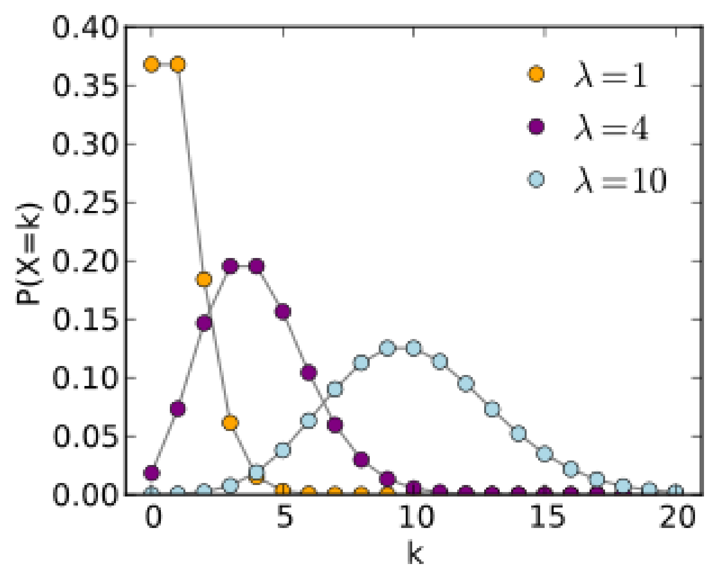
The G(N,p) model:

A given node in the network is connected with independent probability p to each of the N – 1 other nodes.
Thus the probability of being connected to k (and only k) other nodes is $p^k(1-p)^{N-1-k}$
There are ${}^{N-1}C_k$ ways to choose those k other vertices, and hence the total probability of being connected to exactly k others is
$p_k = {}^{N-1}C_k\, p^k(1-p)^{N-1-k}$
which is the Binomial distribution having mean Np and variance $Np(1-p)$

As N becomes large with p being extremely small ($\to 0$), such that $Np = \langle k \rangle = \lambda$ is finite, this tends to the Poisson distribution

$$P(k) = e^{-\lambda}(\lambda^k/k!)$$

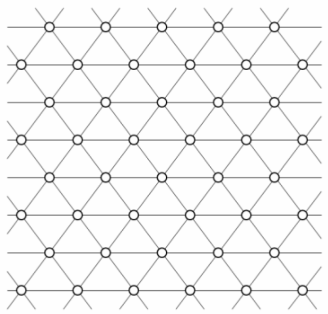
Both the mean and variance is given by λ.
For large values of λ this converges to the bell-shaped Gaussian or Normal distribution



Source: wikipedia

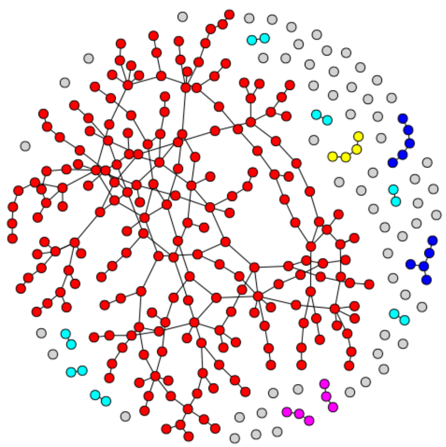
Theoretical understanding of networks

- Regular lattice or grid (*Physics*)
  - average path length ~ *N* (no. of nodes)
  - clustering *high*
  - delta function distribution of degree (links/node)



- Random networks (*Graph theory*)

  Also known as Erdos-Renyi networks
  - average path length ~ log *N*
  - clustering low
  - Poisson distribution of degree



Empirical networks are not random – many have certain structural patterns

Macro-patterns

"It's a *small world*": The Milgram Experiment

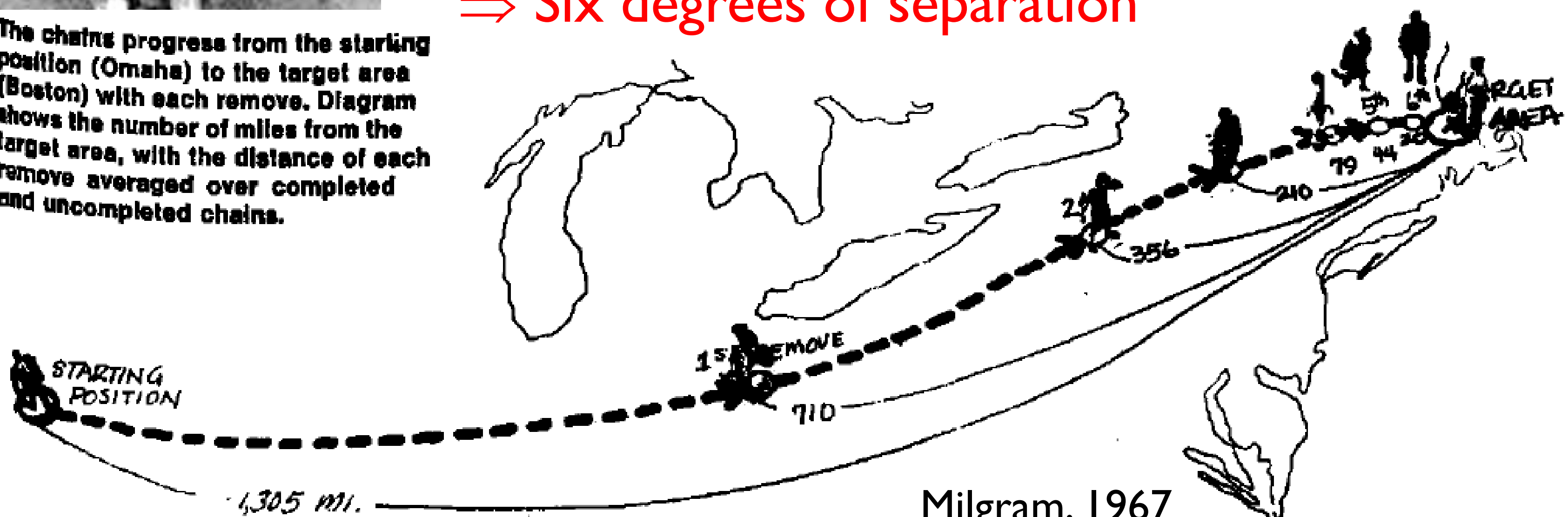
eweb.furman.edu

Stanley Milgram (1933-1984), US social psychologist

Arbitrarily selected individuals in Nebraska were asked to generate acquaintance chains (knowing on first name basis) connecting them to a target individual in Boston

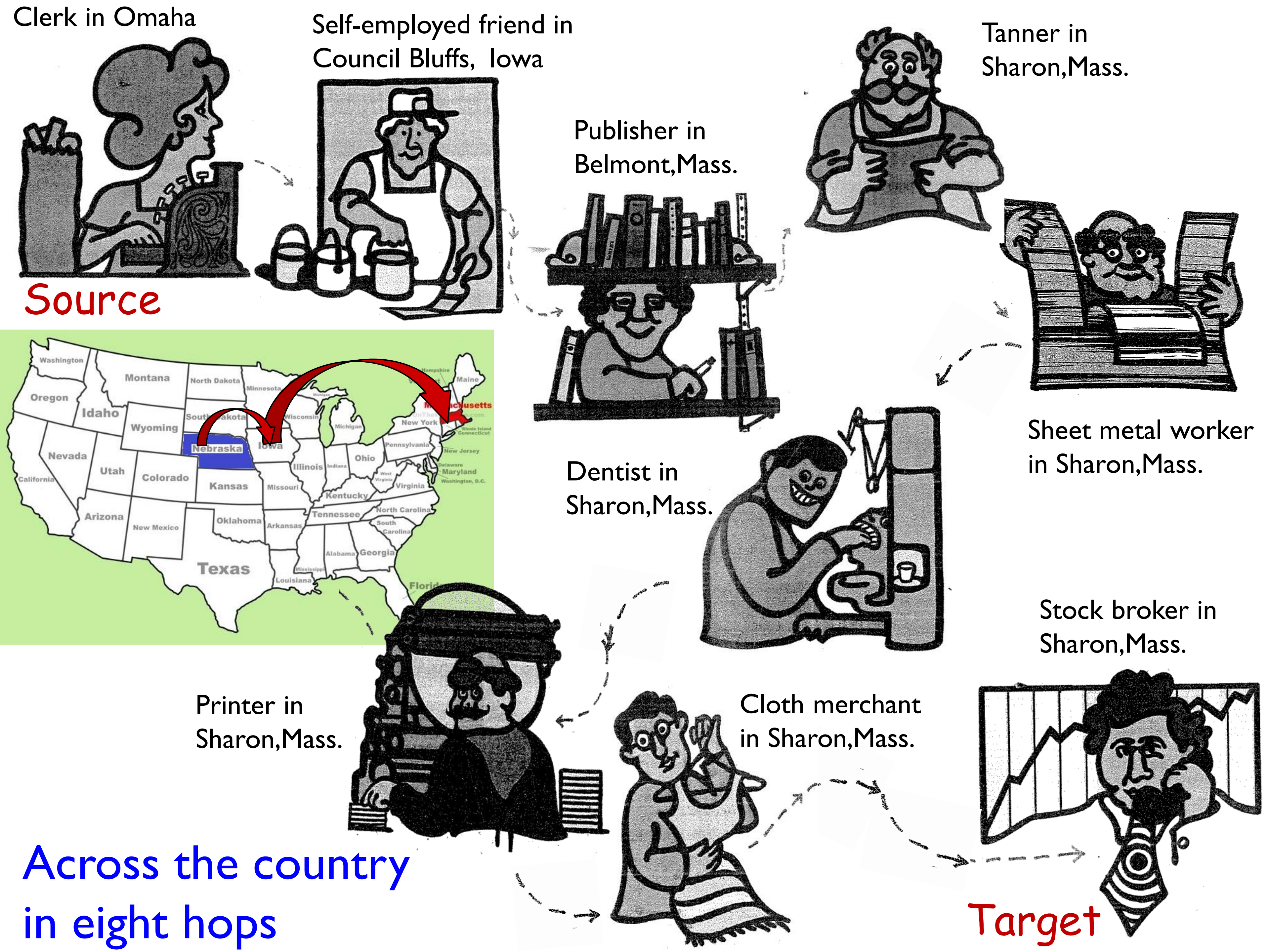
In one experiment, 64 of the 296 chains initiated eventually reached the target – the mean number of intermediaries between source and target being slightly larger than 5

$\Rightarrow$ Six degrees of separation



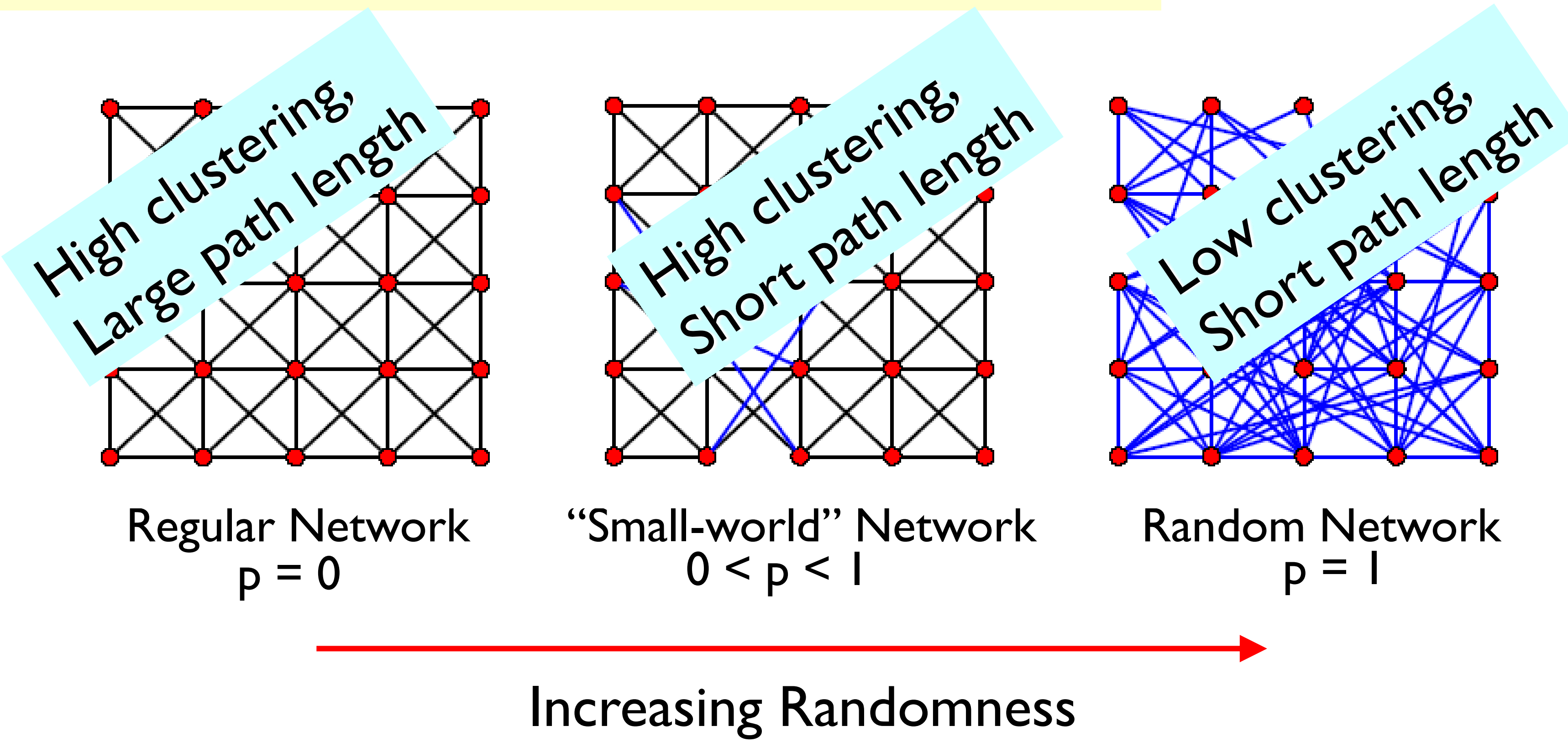
The chains progress from the starting position (Omaha) to the target area (Boston) with each remove. Diagram shows the number of miles from the target area, with the distance of each remove averaged over completed and uncompleted chains.

Milgram, 1967



Clerk in Omaha
Source
Self-employed friend in Council Bluffs, Iowa
Publisher in Belmont,Mass.
Tanner in Sharon,Mass.
Sheet metal worker in Sharon,Mass.
Dentist in Sharon,Mass.
Stock broker in Sharon,Mass.
Cloth merchant in Sharon,Mass.
Printer in Sharon,Mass.
Target
Across the country in eight hops
Washington
Montana
North Dakota
Minnesota
Oregon
Idaho
Wyoming
Nebraska
Iowa
Wisconsin
Michigan
New York
Maine
Nevada
Utah
Colorado
Kansas
Missouri
Illinois
Ohio
Pennsylvania
California
Kentucky
Virginia
Maryland
Arizona
New Mexico
Oklahoma
Arkansas
Tennessee
North Carolina
Texas
Alabama
Georgia
Louisiana

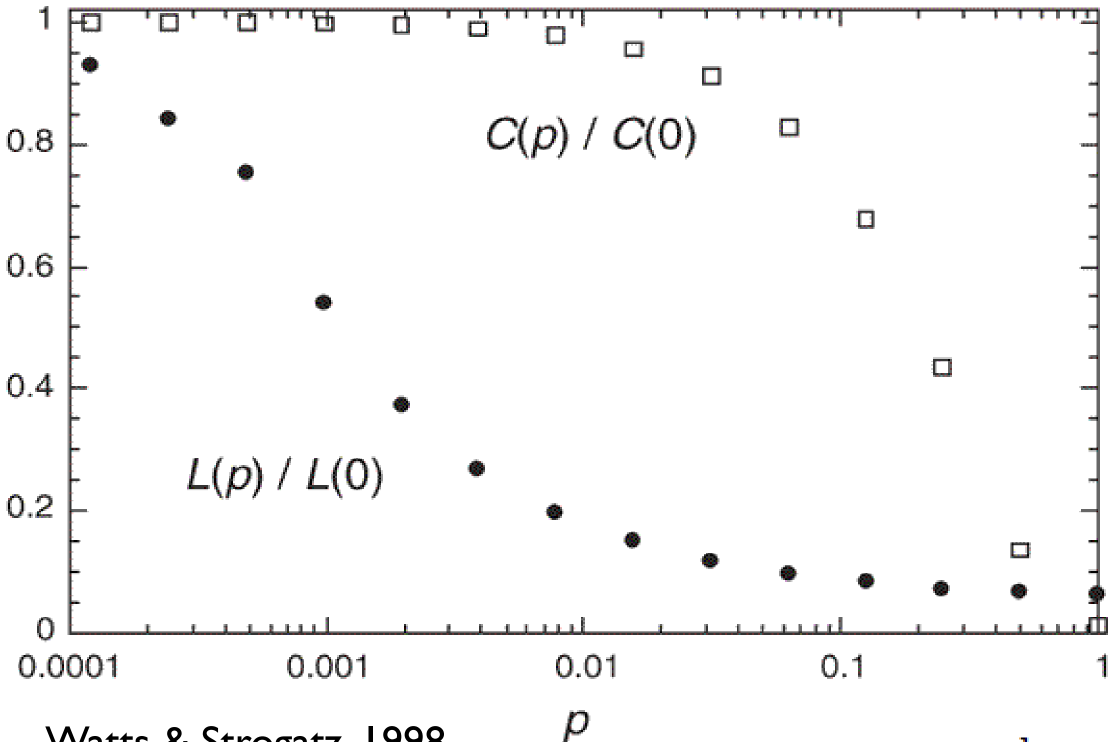
"Small world" networks



p: fraction of random, long-range connections

Watts and Strogatz (1998): Many biological, technological and social networks have connection topologies that lie between the two extremes of completely regular and completely random.

"Small world": Local properties of regular networks but global properties of random networks



Watts & Strogatz, 1998

Characteristic path length

$$L = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}$$

d_{ij} : shortest distance between nodes i and j

Alternatively $$\ell^{-1} = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}^{-1}$$

Clustering coefficient

$$C = \frac{1}{n} \sum_i C_i$$

where

$$C_i = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered on vertex } i}$$

Alternatively $$C = \frac{3 \times \text{ number of triangles in the network}}{\text{number of connected triples of vertices}}$$

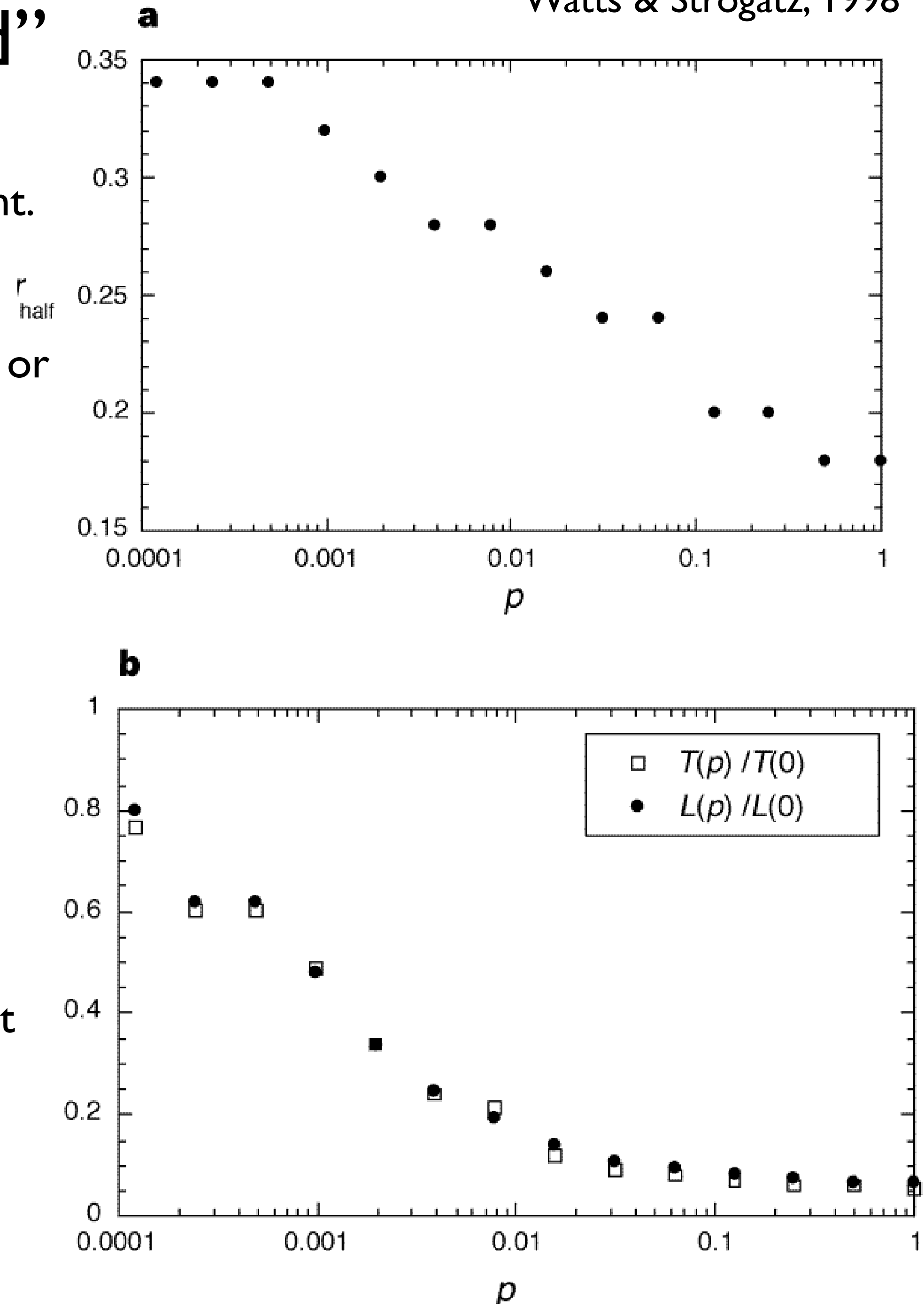
Small-world networks can be highly clustered (like regular networks), yet have small characteristic path lengths (as in random networks).

Epidemics on “Small world”

Watts & Strogatz, 1998

Dynamical process:

- Time t = 0: single infected individual present.
- Each infected agent can infect any of its neighbours with probability r.
- Infected individuals removed (by immunity or death) after unit period of sickness.



Key Results:

- Critical infectiousness r_{half}, at which the disease infects half the population, decreases with p
- Time required for a maximally infectious disease (r = 1) to spread throughout the entire population $T(p)$ has same form as characteristic path length $L(p)$

⇒ rewiring only a few links in the original lattice causes global infection to occur almost as fast as in random network

Implication:

“Control the truck-drivers”

Do *small-world* networks occur in real life ?

	# nodes	Avg degree	Avg path length		Clustering coefficient	
Network	Size	$\langle k \rangle$	ℓ	ℓ_{rand}	C	C_{rand}
WWW, site level, undir.	153 127	35.21	3.1	3.35	0.1078	0.00023
Internet, domain level	3015–6209	3.52–4.11	3.7–3.76	6.36–6.18	0.18–0.3	0.001
Movie actors	225 226	61	3.65	2.99	0.79	0.00027
LANL co-authorship	52 909	9.7	5.9	4.79	0.43	1.8×10^{-4}
MEDLINE co-authorship	1 520 251	18.1	4.6	4.91	0.066	1.1×10^{-5}
SPIRES co-authorship	56 627	173	4.0	2.12	0.726	0.003
NCSTRL co-authorship	11 994	3.59	9.7	7.34	0.496	3×10^{-4}
Math. co-authorship	70 975	3.9	9.5	8.2	0.59	5.4×10^{-5}
Neurosci. co-authorship	209 293	11.5	6	5.01	0.76	5.5×10^{-5}
E. coli, substrate graph	282	7.35	2.9	3.04	0.32	0.026
E. coli, reaction graph	315	28.3	2.62	1.98	0.59	0.09
Ythan estuary food web	134	8.7	2.43	2.26	0.22	0.06
Silwood Park food web	154	4.75	3.40	3.23	0.15	0.03
Words, co-occurrence	460.902	70.13	2.67	3.03	0.437	0.0001
Words, synonyms	22 311	13.48	4.5	3.84	0.7	0.0006
Power grid	4941	2.67	18.7	12.4	0.08	0.005
C. Elegans	282	14	2.65	2.25	0.28	0.05

Albert & Barabasi, 2003