Non-equilibrium stationary states of run and tumble particles in confining potentials

Grégory Schehr
Laboratoire de Physique Théorique et Hautes Energies,
CNRS - Sorbonne Université

Frontiers in Statistical Physics, 75th birthday
of RRI, Bangalore, 04-08 Dec. 2023

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- Pierre Le Doussal (Ecole Normale Supérieure, Paris)
- Satya N. Majumdar (LPTMS, University Paris-Saclay)
- Alberto Rosso (LPTMS, University Paris-Saclay)
- Sanjib Sabhapandit (RRI, Bangalore)
- Léo Touzo (Ecole Normale Supérieure, Paris)

A simple model of active particle in d=1

- The free run and tumble particle (RTP) on the line

$$X(t=0) = X_0 \quad , \quad \frac{dX}{dt} = v_0 \, \sigma(t)$$

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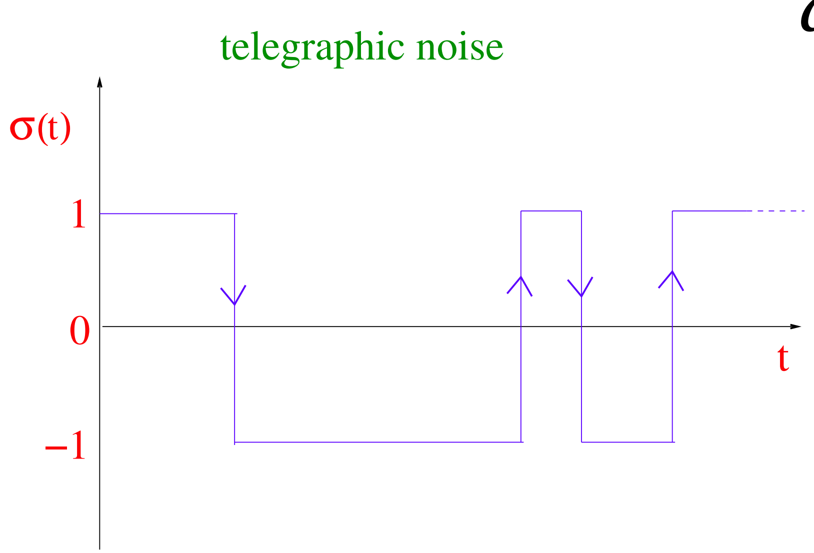

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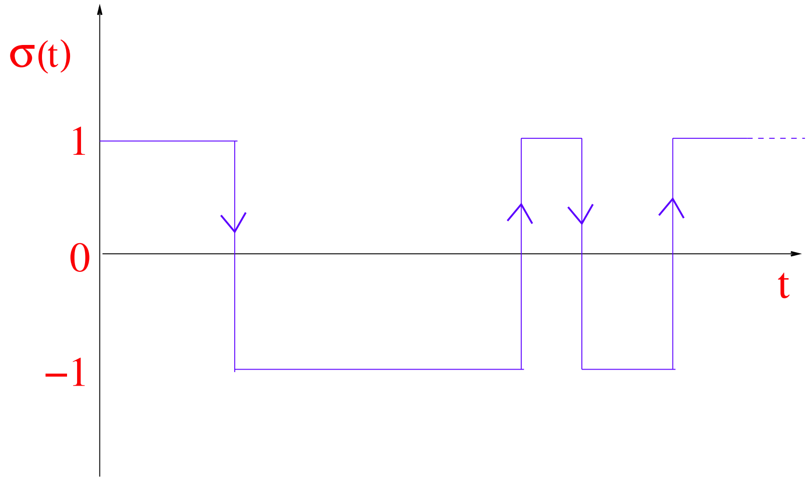

$$\sigma(t+dt) = \begin{cases} -\sigma(t) & \text{with proba .} \quad \gamma\, dt \\ \sigma(t) & \text{with proba .} \quad 1 - \gamma\, dt \end{cases}$$

$$\langle \sigma(t_1)\sigma(t_2)\rangle = e^{-2\gamma|t_1-t_2|}$$

→ $X(t)$ is non-Markovian

A useful model of active particle in d=1

- Free RTP on the line

$$\frac{dX}{dt} = v_0\,\sigma(t)$$

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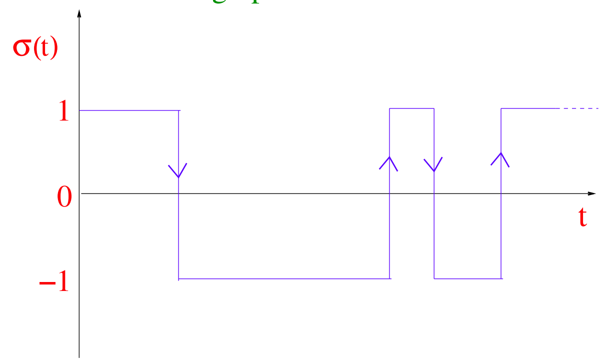

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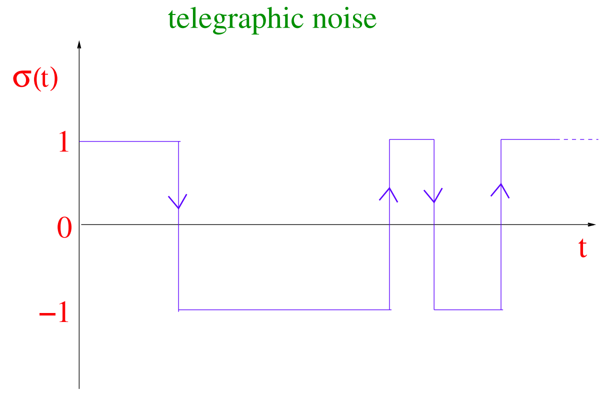

- The single free RTP (or persistent random walk) has already a long story

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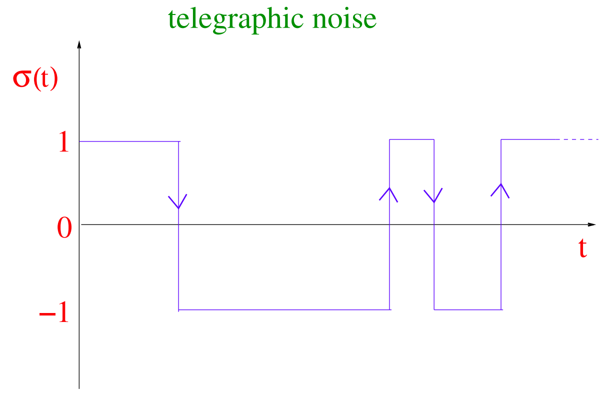


- The single free RTP (or persistent random walk) has already a long story

▹ R. Fürth (1920) ``The Brownian motion when considering persistence of the direction of movement. With applications to the movement of living infusoria"

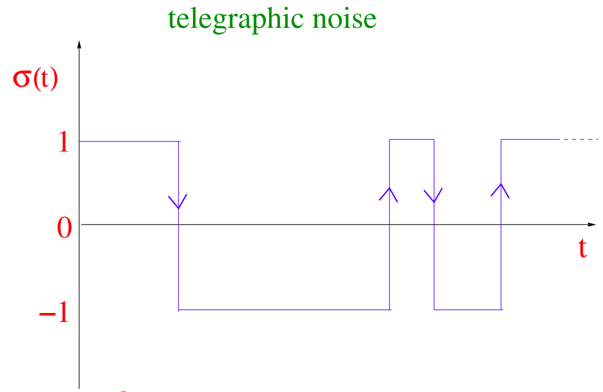
▹ M. Kac (1974), ``A stochastic model related to the telegrapher's equation"

▹ see also R. P. Feynman (1965), ``Relativistic chessboard model"

A useful model of active particle in d=1

- Free RTP on the line

$$\begin{cases} X(t=0)=0 \quad , \\ \sigma(0) = \pm \qquad \text{w . prob . } 1/2 \end{cases} \qquad \frac{dX}{dt} = v_0\, \sigma(t)$$



- Propagator of the free RTP

Bessel funct.

$$P(x,t) = \frac{e^{-\gamma t}}{2}\left[\delta(x - v_0 t) + \delta(x + v_0 t) + \frac{\gamma}{v_0}\left(I_0(\rho) + \frac{\gamma t\, I_1(\rho)}{\rho}\right)\Theta(v_0 t - |x|)\right]$$

where $\rho = \dfrac{\gamma}{v_0}\sqrt{v_0^2 t^2 - x^2}$

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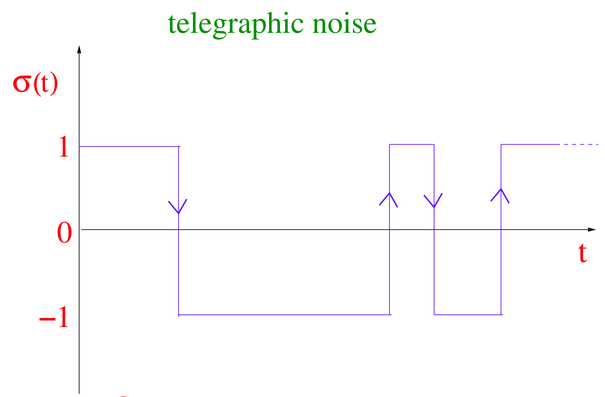

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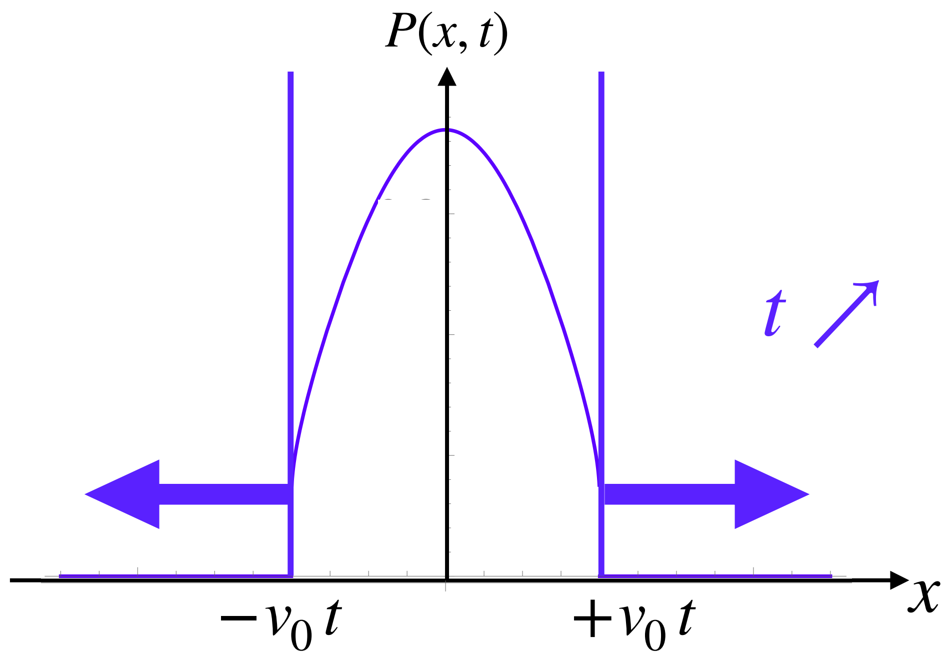

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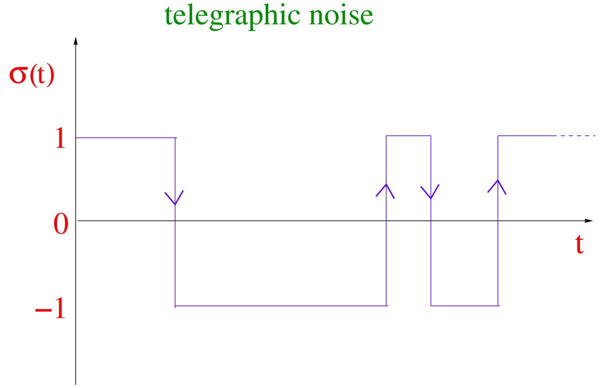

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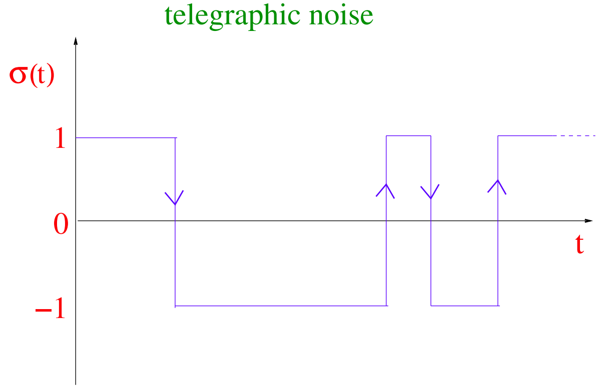
- The Brownian limit is recovered in the scaling limit $\gamma \to \infty, v_0 \to \infty$ keeping $v_0^2/(2\gamma) = D_0$ fixed, i.e.,

$$P(x,t) \longrightarrow \frac{e^{-\frac{x^2}{4D_0 t}}}{\sqrt{4\pi D_0 t}}$$

A useful model of active particle in d=1



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- This talk:

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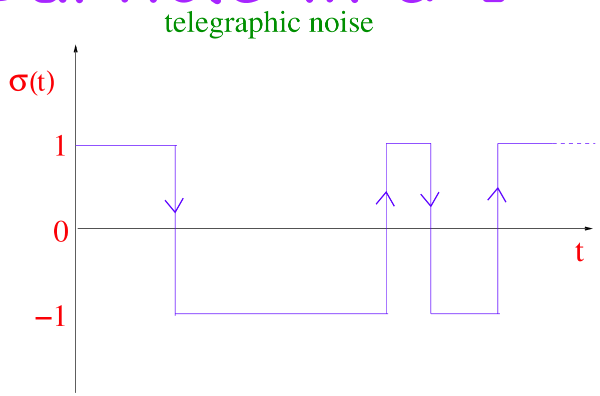

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- This talk:

Q1: what happens in the presence of an external potential $V(x)$?

Q2: what are the effects of interactions ?

Outline

- Two states RTP: stationary state in a confining potential $V(x)$

- Two particles ($N = 2$) with attractive interaction

- Many RTP's in interaction: the active Dyson Brownian motion

- Conclusion

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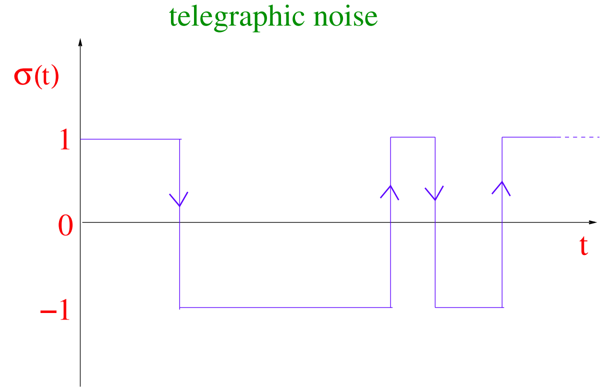
A single RTP in an external potential

- The case of a confining potential $V(x) = \alpha |x|^p \quad , \quad \alpha > 0 \quad \& \quad p > 0$

$$\frac{dX}{dt} = f(x) + v_0 \sigma(t)$$

$$f(x) = -V'(x)$$



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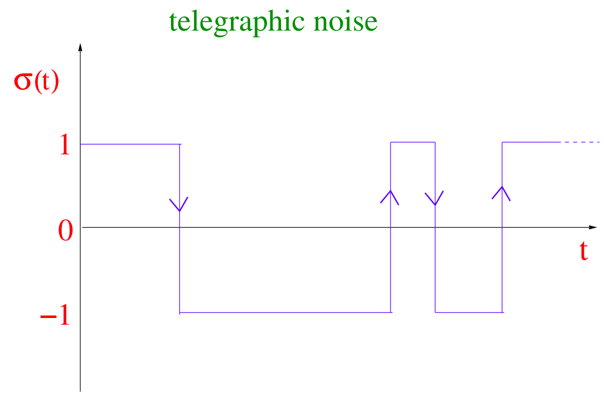

- Fixed points of the dynamics

$$0 = f(x_+) + v_0$$ Fixed point in the $\sigma = +$ state

$$0 = f(x_-) - v_0$$ Fixed point in the $\sigma = -$ state

A single RTP in an external potential

- A first graphical approach for $V(x) = \alpha\,|\,x\,|^p$

$$\frac{dX}{dt} = f(x) + v_0\,\sigma(t)$$

$$p > 1$$

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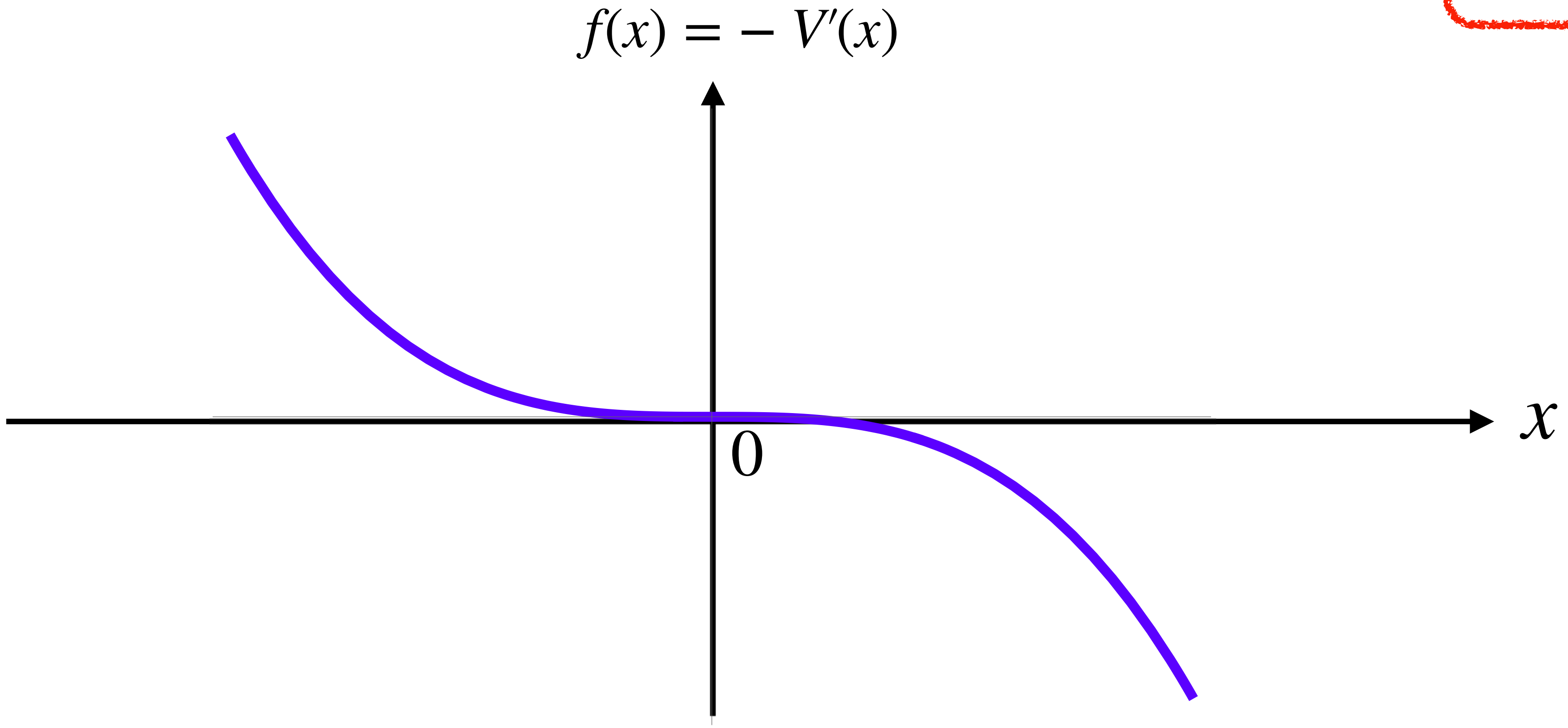

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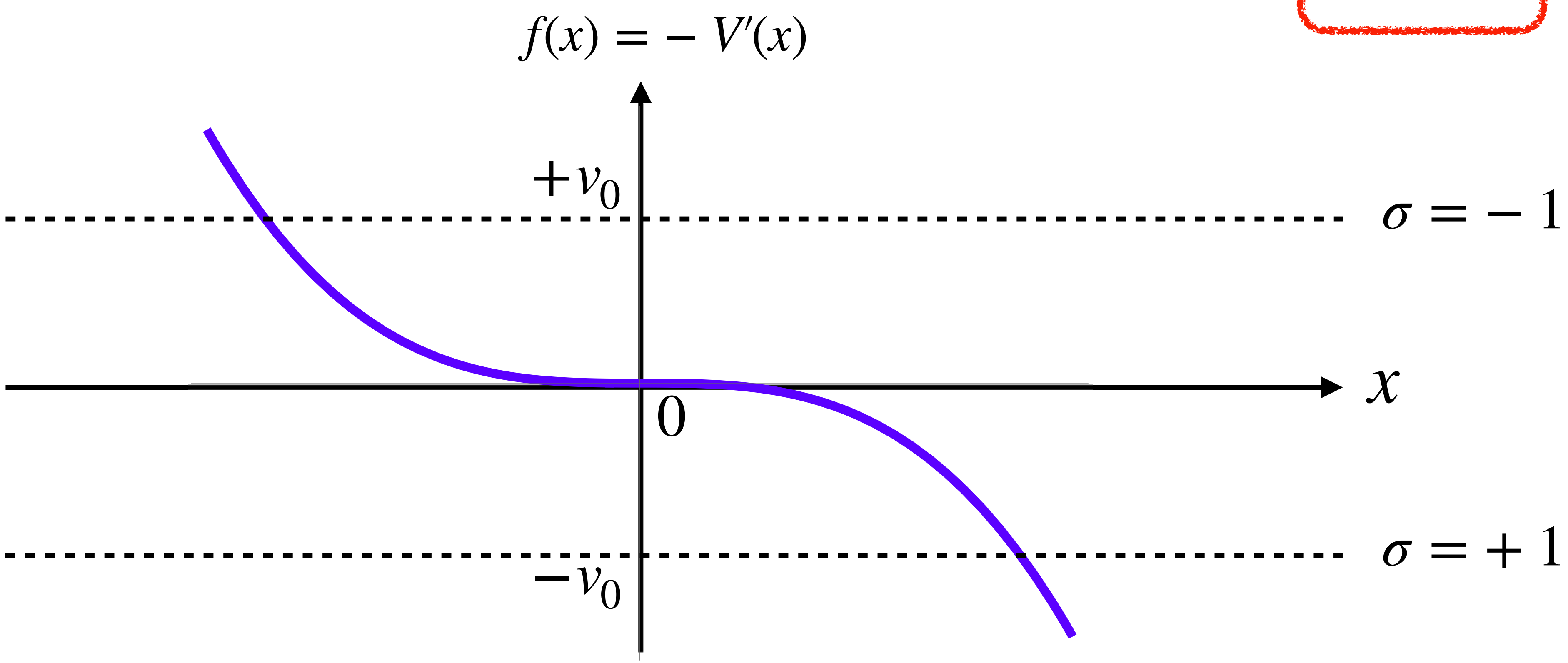

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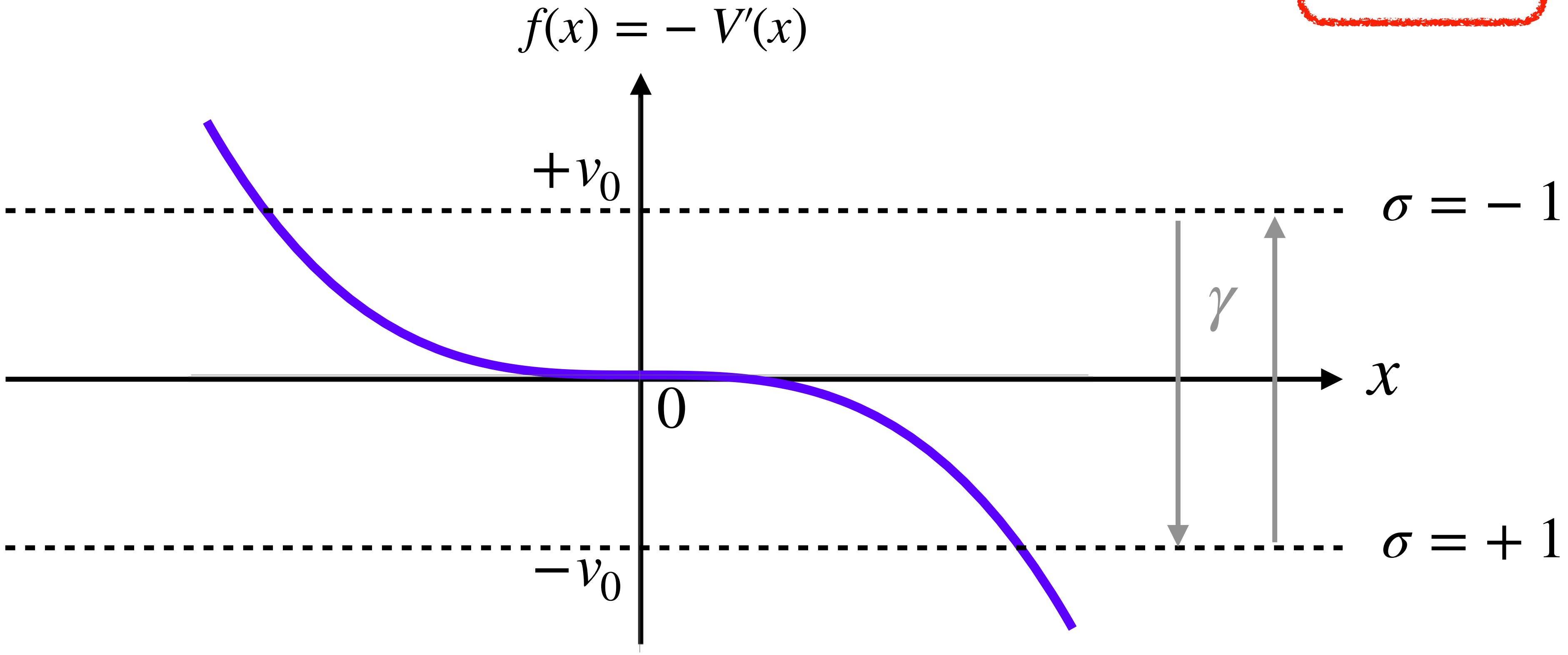

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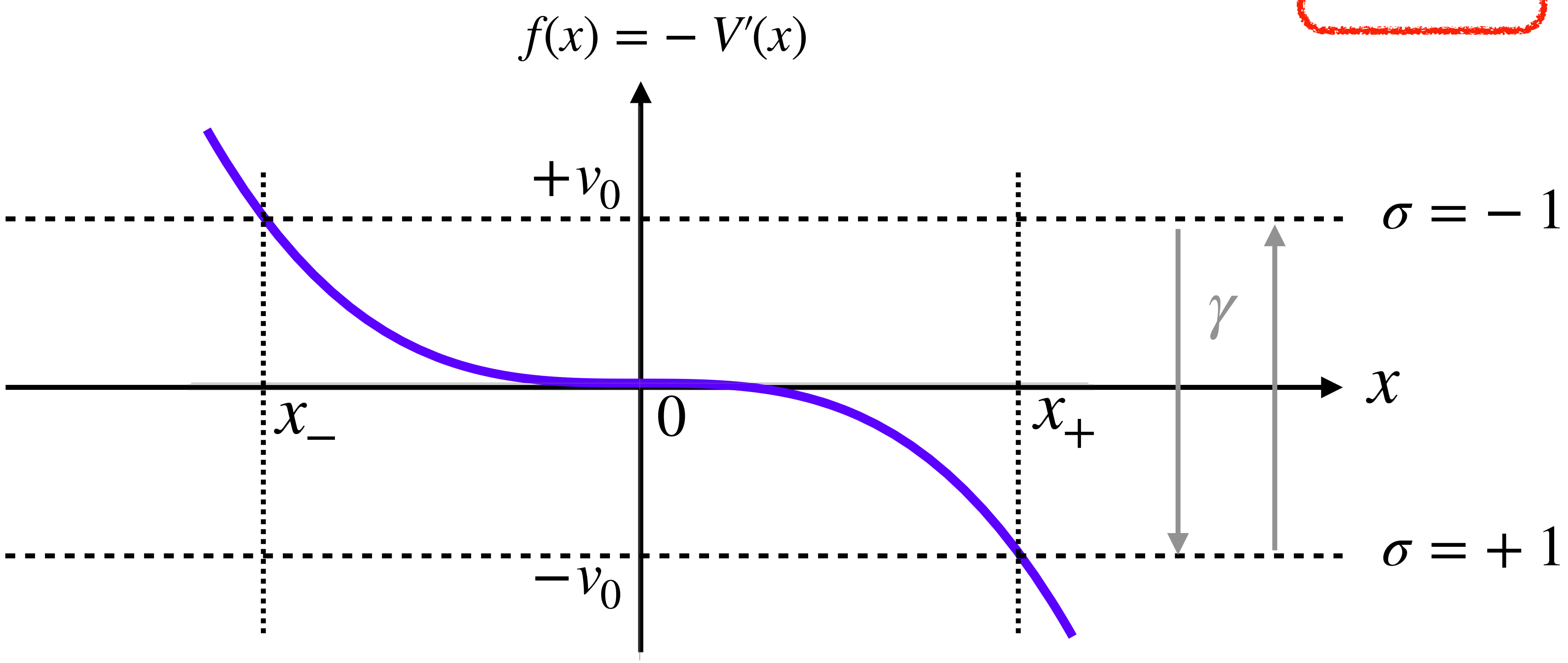

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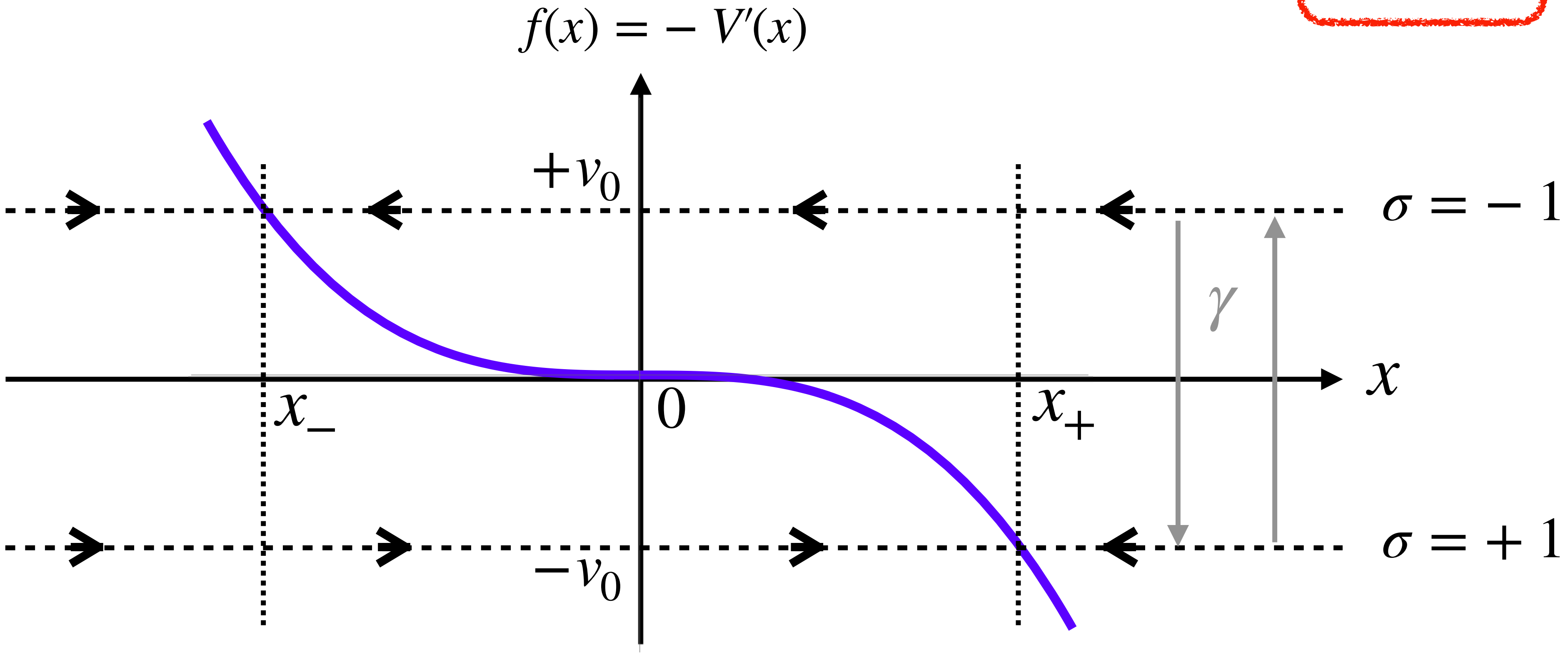

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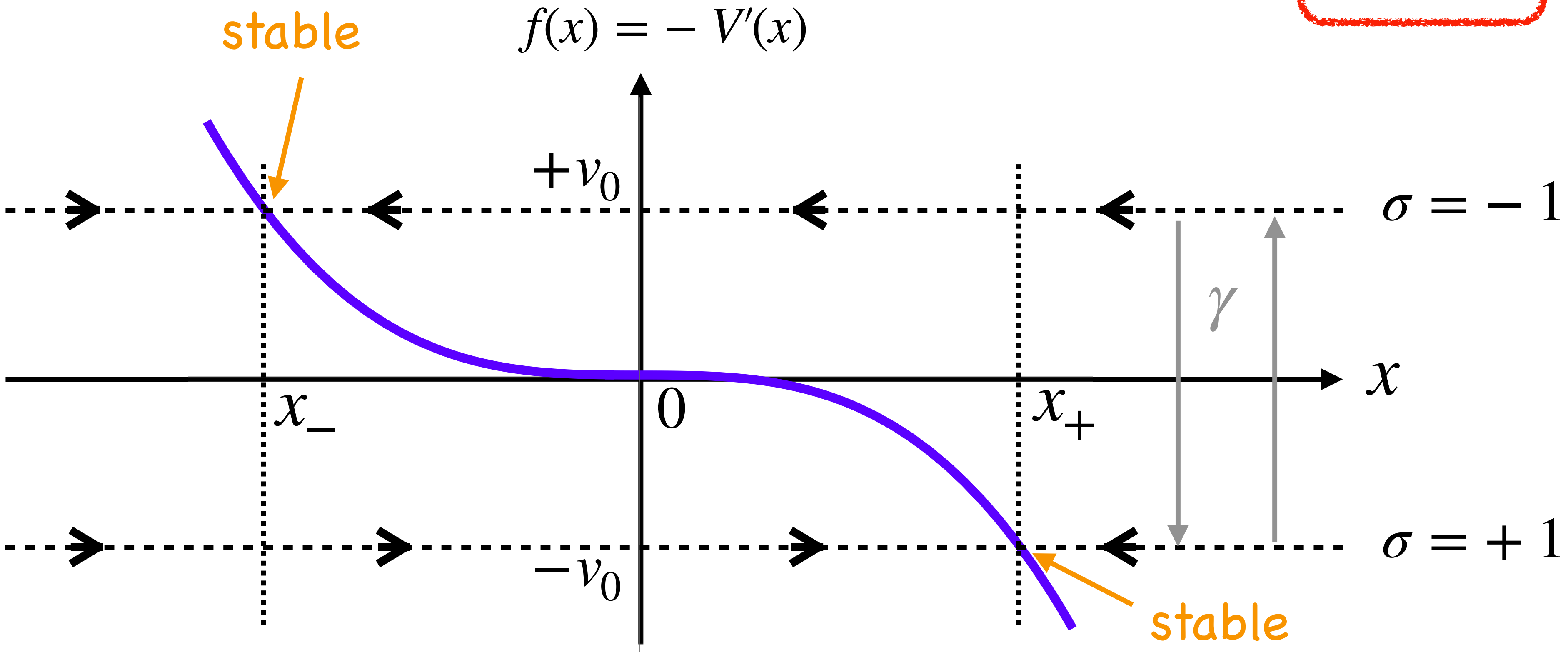

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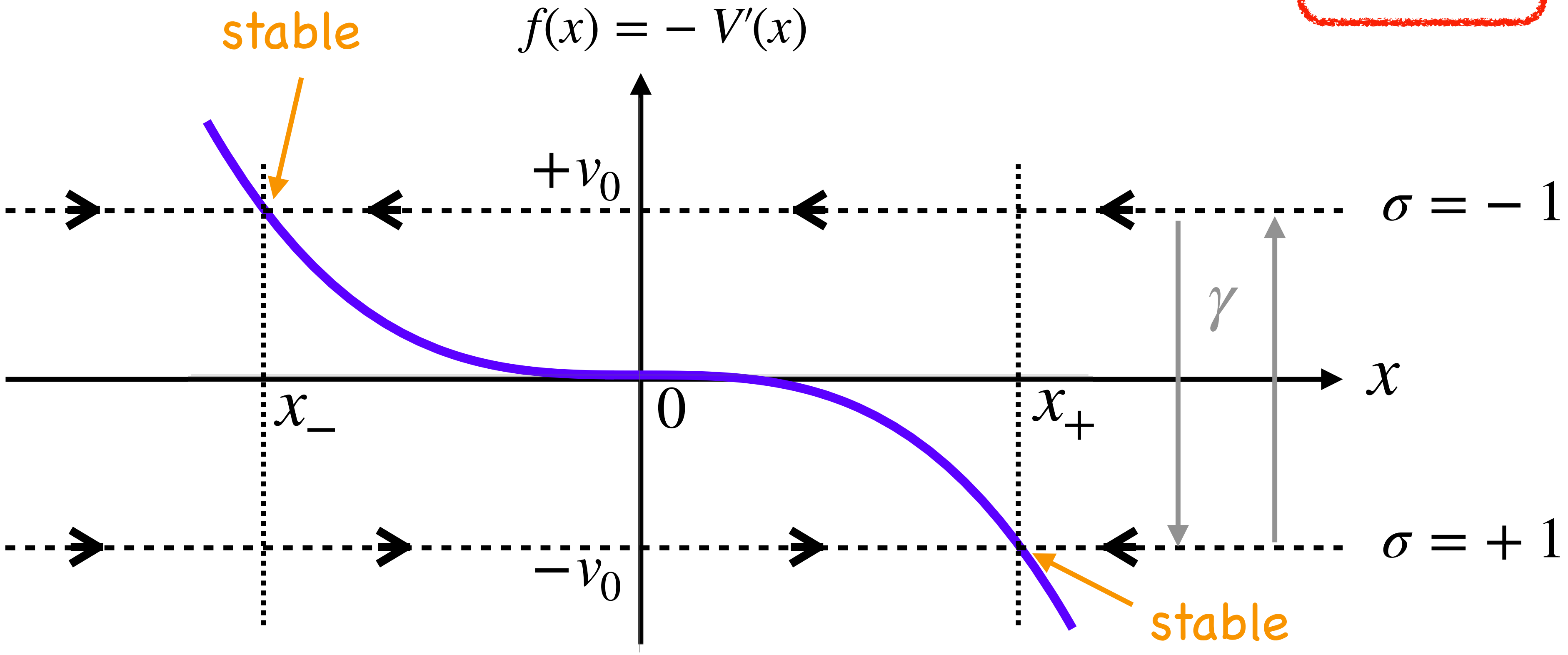

The particle gets trapped in $[x_-, x_+]$ after a finite time

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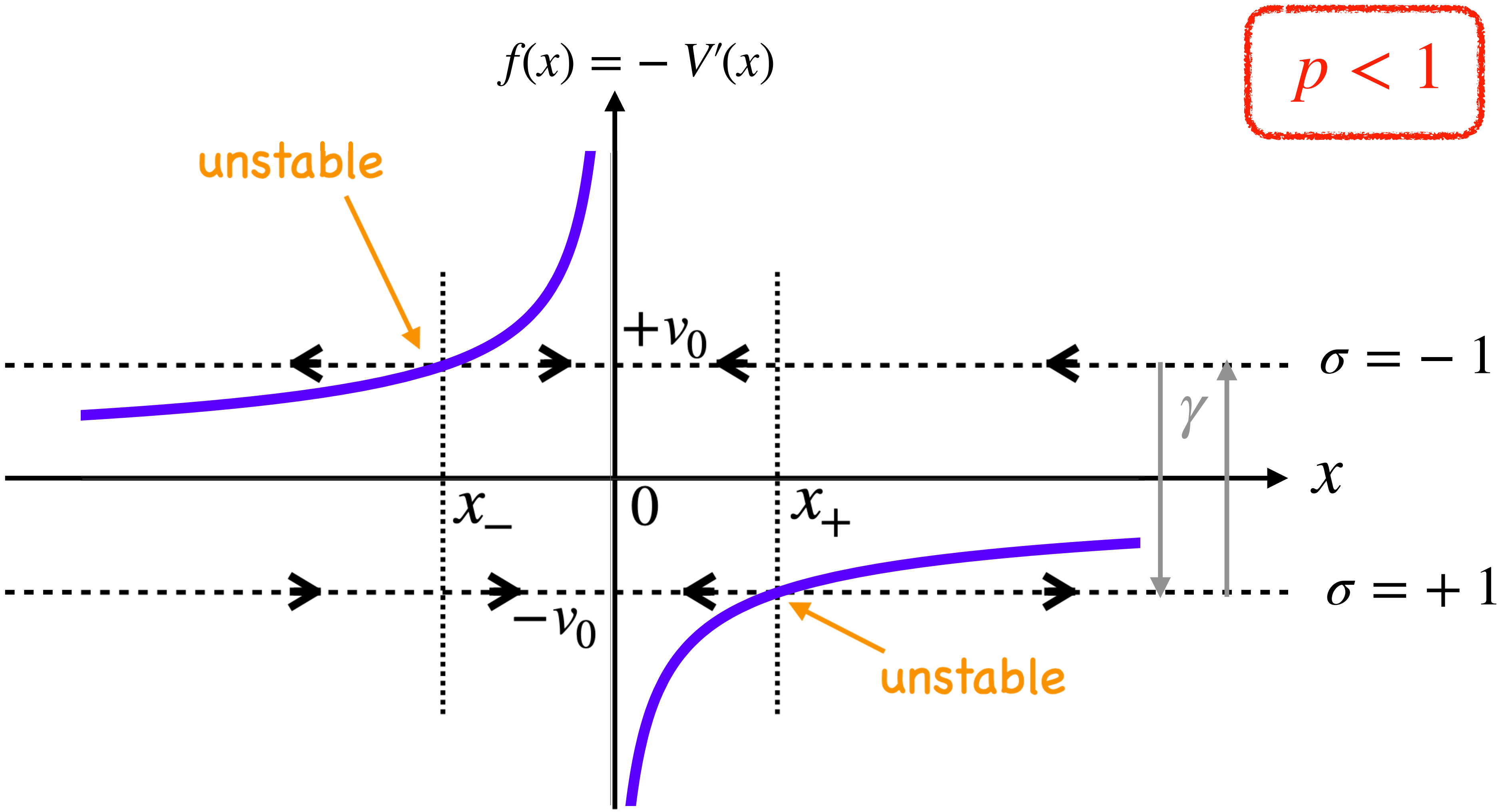

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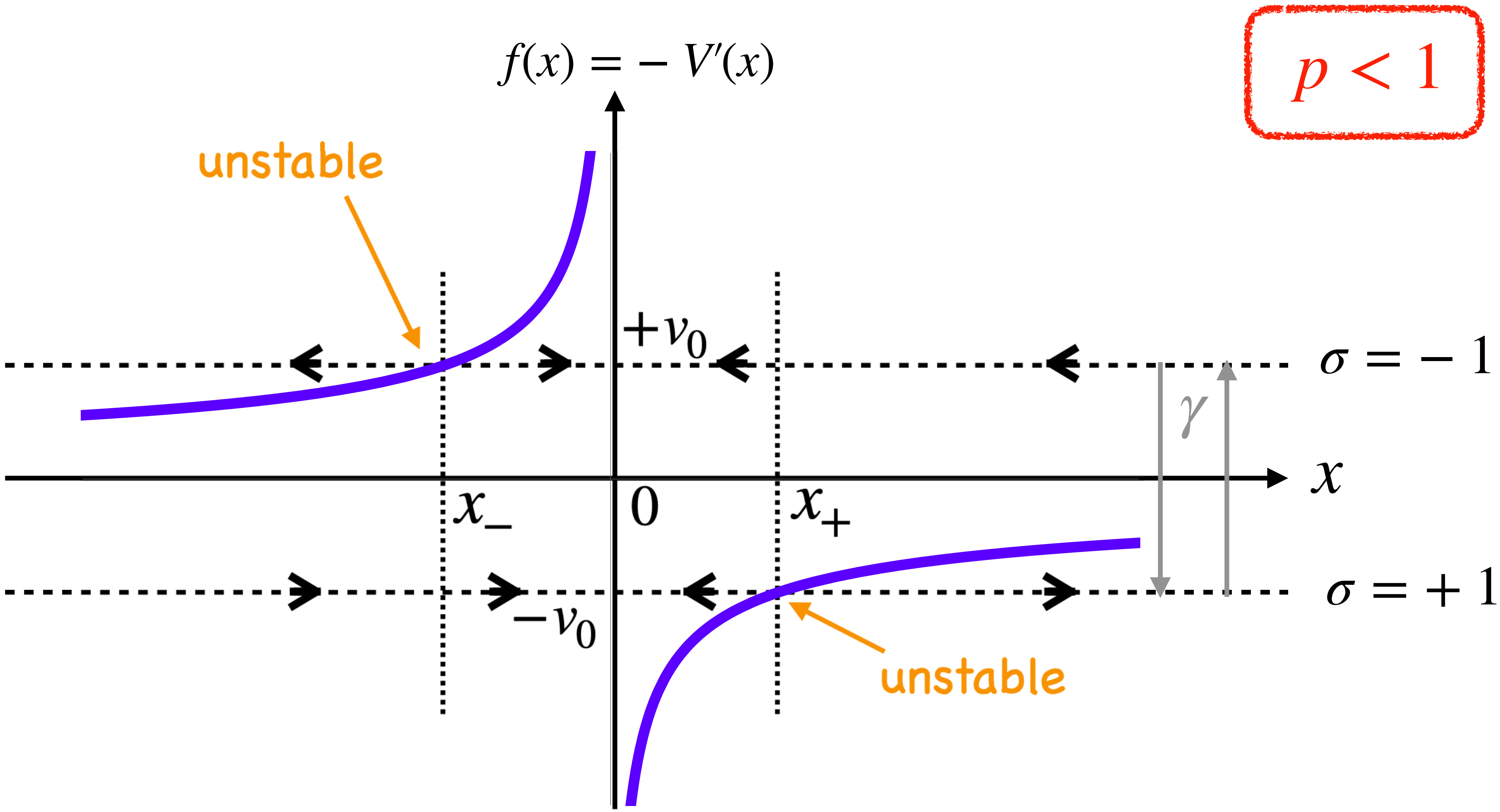

The particle will end up at $x = 0$

A single RTP in an external potential

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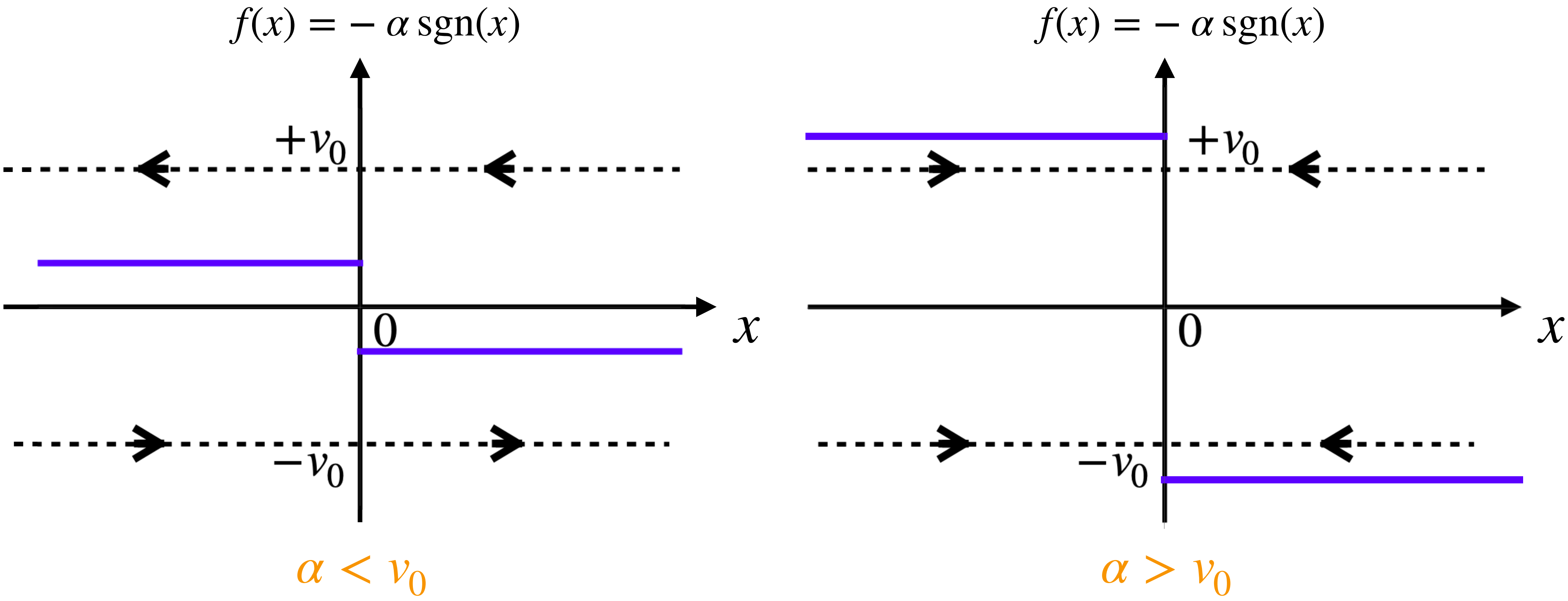
$p = 1$



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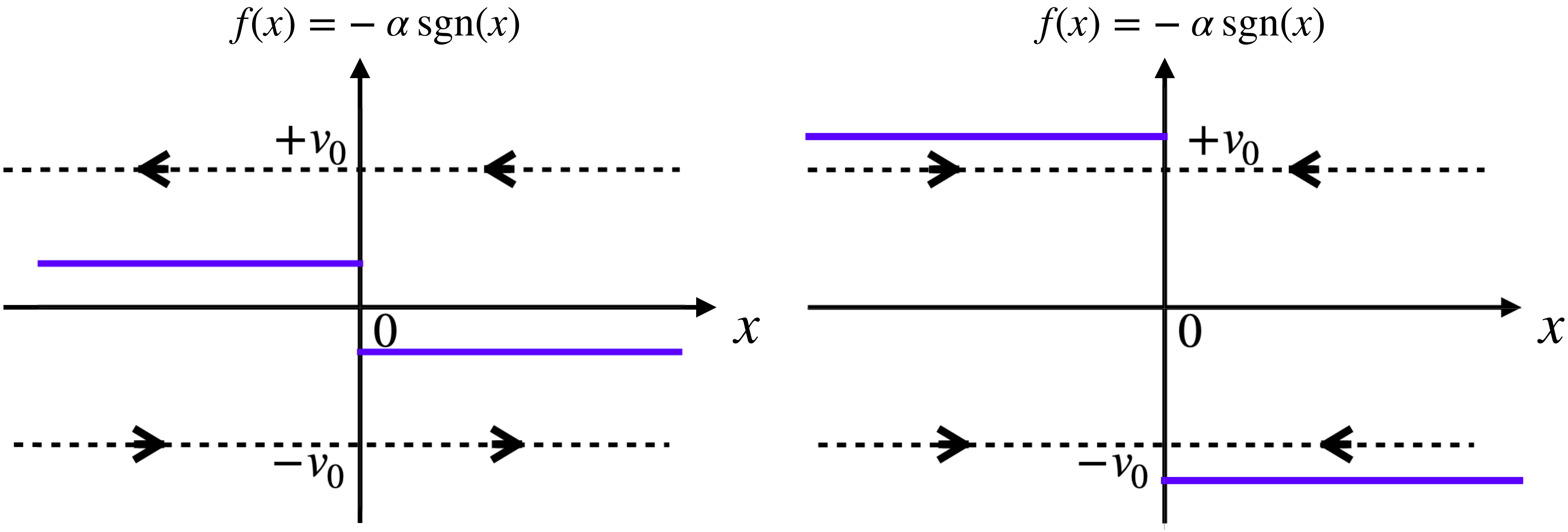

$\alpha < v_0$ $\alpha > v_0$

→ One expects a transition at $\alpha_c = v_0$

A single RTP in an external potential

- Probability densities of the particle 's position $P_\pm(x,t)$

$P_+(x,t)\,dx =$ Proba. to find the particle in $[x, x+dx]$ in the state $\sigma = +$ at time t

$P_-(x,t)\,dx =$ Proba. to find the particle in $[x, x+dx]$ in the state $\sigma = -$ at time t

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- Exact solution via coupled Fokker-Planck equations

$$\frac{\partial P_+}{\partial t} = -\frac{\partial}{\partial x}\left[(f(x)+v_0)P_+\right] - \gamma P_+ + \gamma P_-$$

$$\frac{\partial P_-}{\partial t} = -\frac{\partial}{\partial x}\left[(f(x)-v_0)P_-\right] + \gamma P_+ - \gamma P_-$$

+ initial and boundary conditions (to be specified later)

A single RTP in an external potential

$$\frac{d}{dx}\left[(f(x)+v_0)P_+\right]+\gamma P_+-\gamma P_-=0$$

$$\frac{d}{dx}\left[(f(x)-v_0)P_-\right]-\gamma P_++\gamma P_-=0$$

⇒ two boundary conditions need to be specified

A single RTP in an external potential

- Stationary state solutions $P_{\pm}(x) = \lim_{t \to \infty} P_{\pm}(x, t)$

$$\frac{d}{dx}\left[(f(x) + v_0)P_+\right] + \gamma P_+ - \gamma P_- = 0$$

$$\frac{d}{dx}\left[(f(x) - v_0)P_-\right] - \gamma P_+ + \gamma P_- = 0$$

$\Longrightarrow$ two boundary conditions need to be specified

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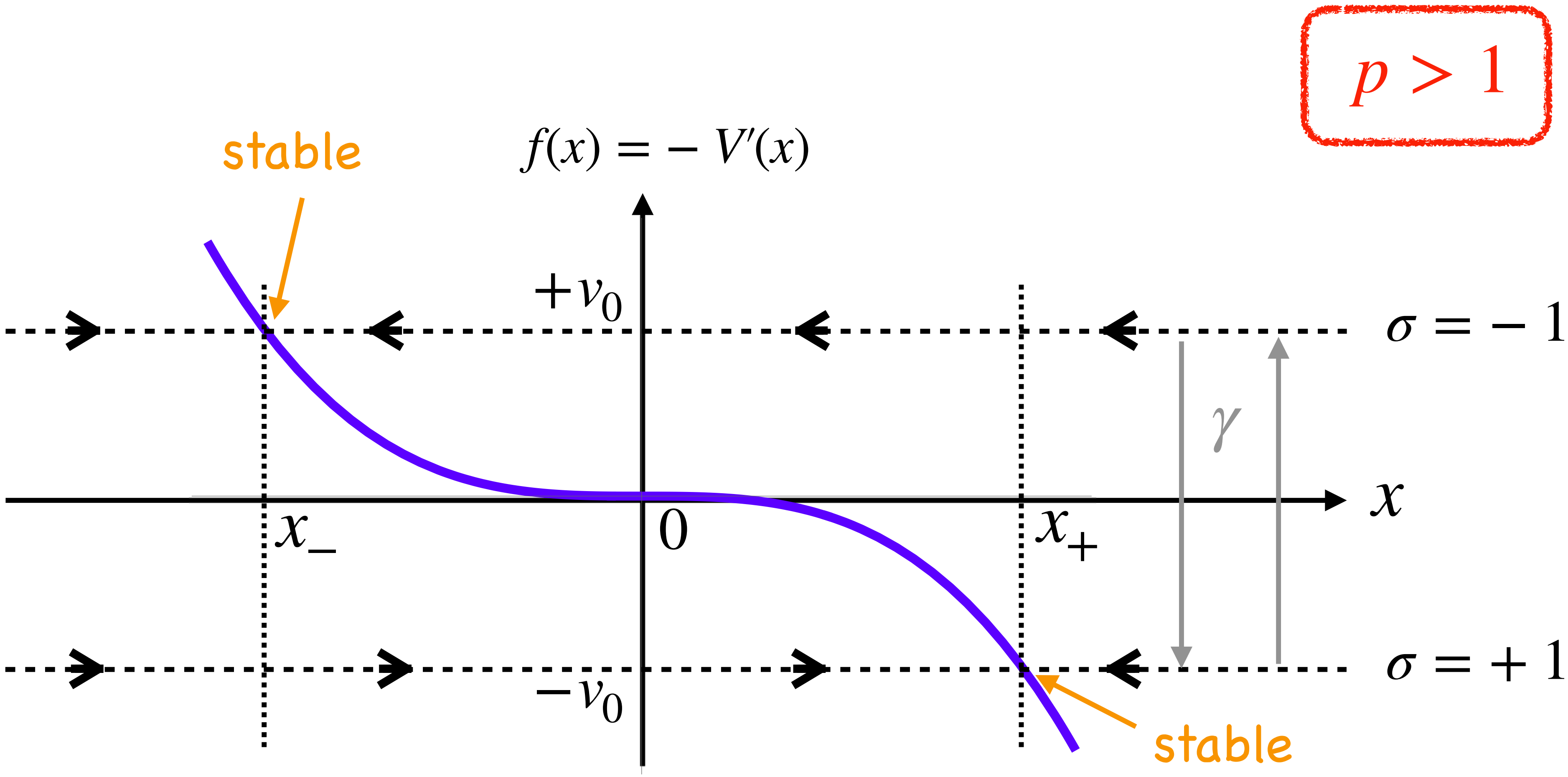

The particle gets trapped in $[x_-, x_+]$ after a finite time

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$$\frac{d}{dx}\left[(f(x)+v_0)P_+\right] + \gamma P_+ - \gamma P_- = 0 \qquad (1)$$

$$\frac{d}{dx}\left[(f(x)-v_0)P_-\right] - \gamma P_+ + \gamma P_- = 0 \qquad (2)$$

⟹ two boundary conditions need to be specified

- For $p > 1$ the two boundary conditions thus read

$$P_-(x_+) = 0$$

$$P_+(x_-) = 0$$

A single RTP in an external potential

- Stationary state solutions $P_{\pm}(x) = \lim_{t \to \infty} P_{\pm}(x,t)$

$$\frac{d}{dx}\left[(f(x) + v_0)P_+\right] + \gamma P_+ - \gamma P_- = 0 \qquad (1)$$

$$\frac{d}{dx}\left[(f(x) - v_0)P_-\right] - \gamma P_+ + \gamma P_- = 0 \qquad (2)$$

→ two boundary conditions need to be specified

- For $p > 1$ the two boundary conditions thus read

$$P_-(x_+) = 0$$

$$P_+(x_-) = 0$$

- The Eqs. (1) and (2) can then be solved by introducing

$$P(x) = P_+(x) + P_-(x) \qquad , \qquad Q(x) = P_+(x) - P_-(x)$$

A single RTP in an external potential $V(x)$

- A closed equation for $P(x) = P_+(x) + P_-(x)$

$$\frac{d}{dx}\left[(v_0^2 - f^2(x))P(x)\right] - 2\gamma f(x)\, P(x) = 0$$

⇨ a 1st order eq. that can be solved explicitly

- Explicit expression for $P(x) = P_+(x) + P_-(x)$

$$P(x) = \frac{A}{v_0^2 - f^2(x)} \exp\left(2\gamma \int_0^x dy \frac{f(y)}{v_0^2 - f^2(y)}\right) \quad , \quad x \in [x_-, x_+]$$

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⇨ very different from Boltzmann $P(x) \propto e^{-\beta V(x)}$

see also Klyatskin '78, Lefever et al. '80, Van den Broeck & Hänggi '84, Hänggi & Jung '94

Transition from active to passive behavior

$$P(x) = \frac{A}{v_0^2 - f^2(x)} \exp\left(2\gamma \int_0^x dy \frac{f(y)}{v_0^2 - f^2(y)} \right) \quad , \quad x \in [x_-, x_+]$$

- The special case of the harmonic potential $V(x) = \mu x^2/2$

$$P(x) = A \frac{\mu}{v_0} \left[1 - \left(\frac{\mu x}{v_0} \right)^2 \right]^{\phi} \qquad , \qquad \phi = \frac{\gamma}{\mu} - 1$$

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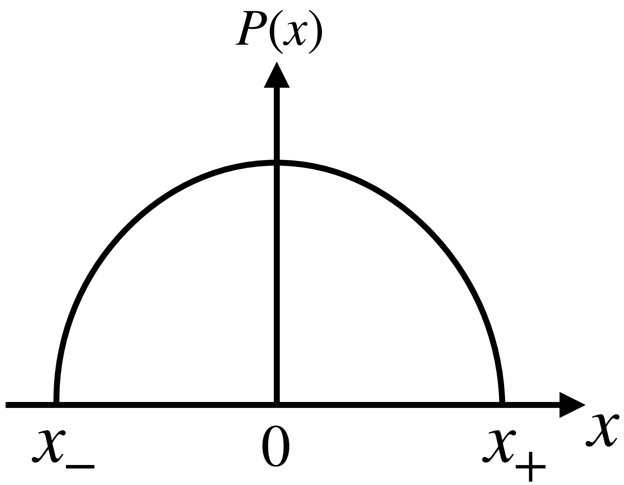
$\gamma > \mu$



« passive »

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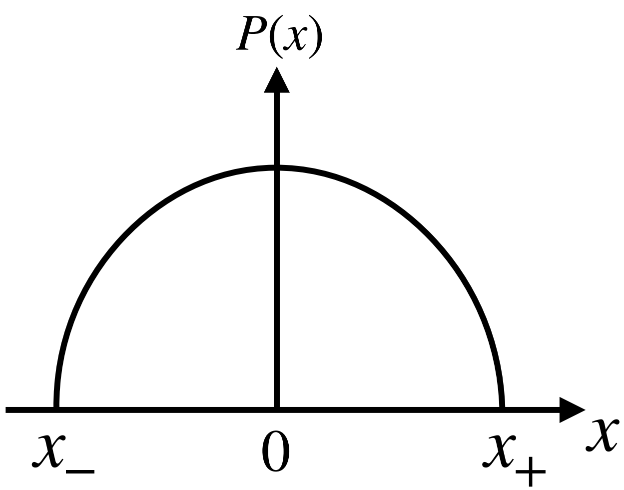
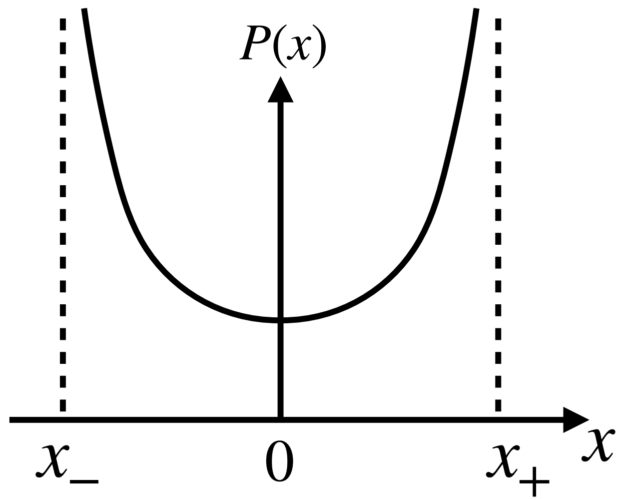
$$P(x) = A\,\frac{\mu}{v_0}\left[1 - \left(\frac{\mu x}{v_0}\right)^2\right]^{\phi} \quad , \quad \phi = \frac{\gamma}{\mu} - 1$$

$\gamma > \mu$



« passive »

$\gamma < \mu$



« active »

Transition from active to passive behavior

$$P(x) = \frac{A}{v_0^2 - f^2(x)} \exp\left(2\gamma \int_0^x dy \frac{f(y)}{v_0^2 - f^2(y)} \right) \quad , \quad x \in [x_-, x_+]$$

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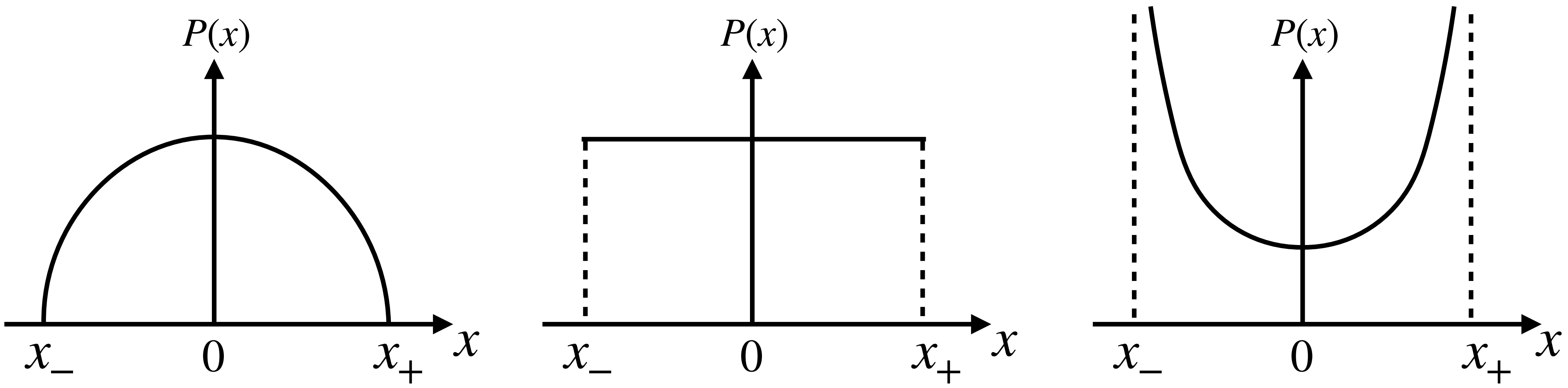
$\gamma > \mu$ $\qquad$ $\gamma = \mu$ $\qquad$ $\gamma < \mu$



« passive » $\qquad$ « active »

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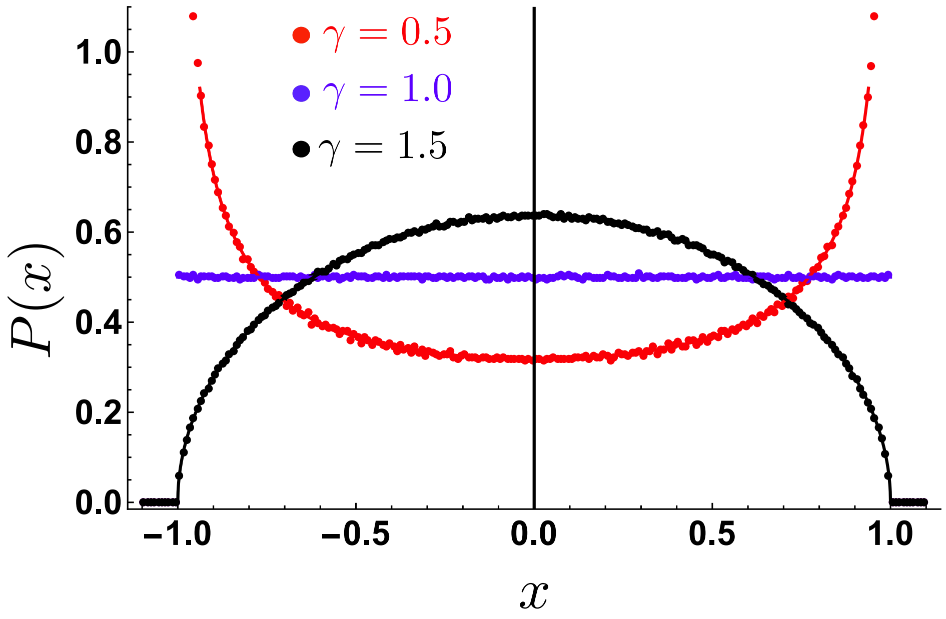

$\mu = 1$

Transition from active to passive behavior

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- The generic case $V(x) = \alpha |x|^p$ with $p > 1$, $\alpha > 0$

Behavior close to the edges:

$$P(x) \propto (x_+ - x)^{\frac{\gamma}{|f'(x_+)|} - 1} \quad , \quad x \to x_+$$

$$P(x) \propto (x - x_-)^{\frac{\gamma}{|f'(x_-)|} - 1} \quad , \quad x \to x_-$$

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change of behavior at $\gamma_c = |f'(x_\pm)|$

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⟹ change of behavior at $\gamma_c = |f'(x_\pm)|$

⟹ shape transition from passive to active

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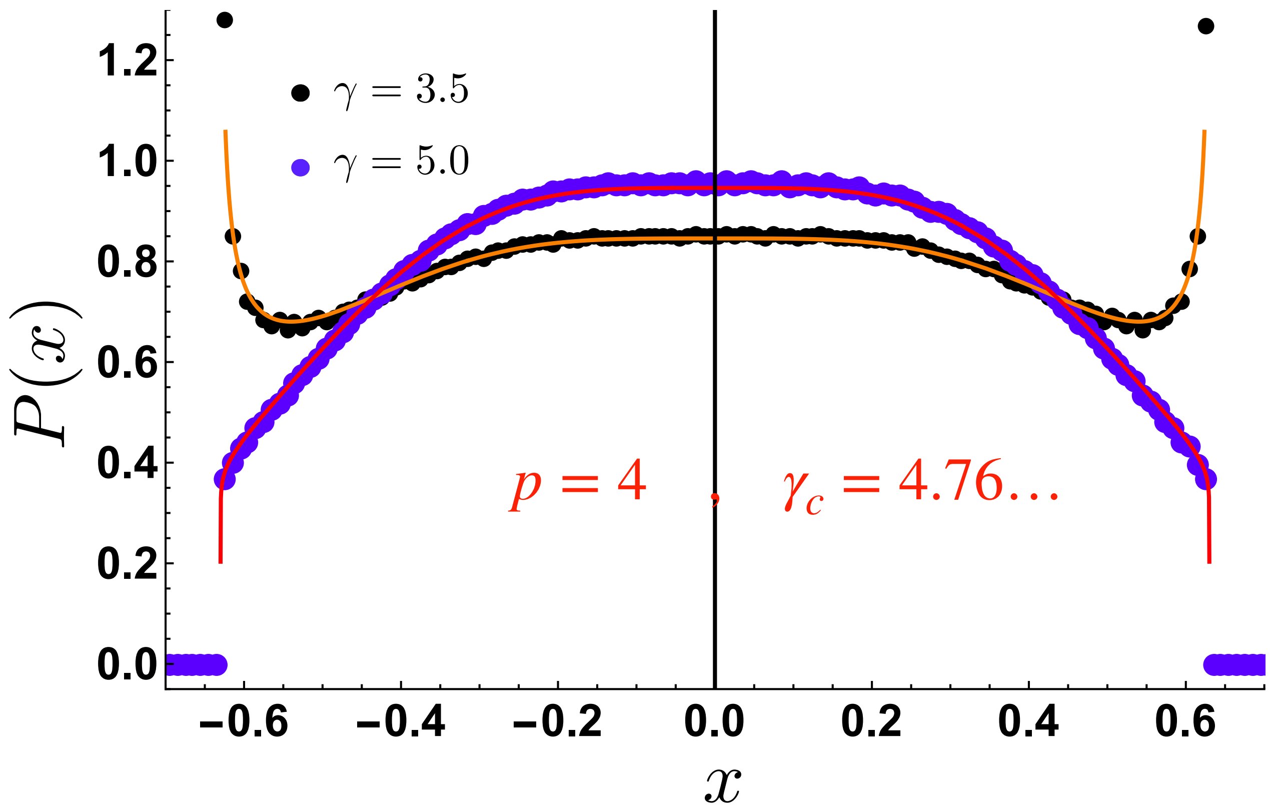
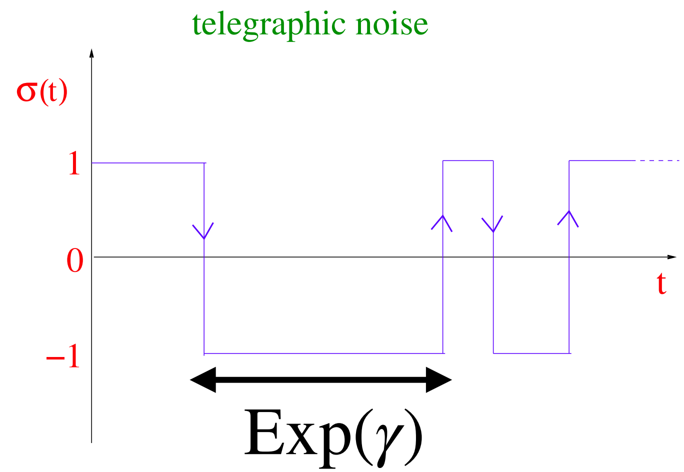


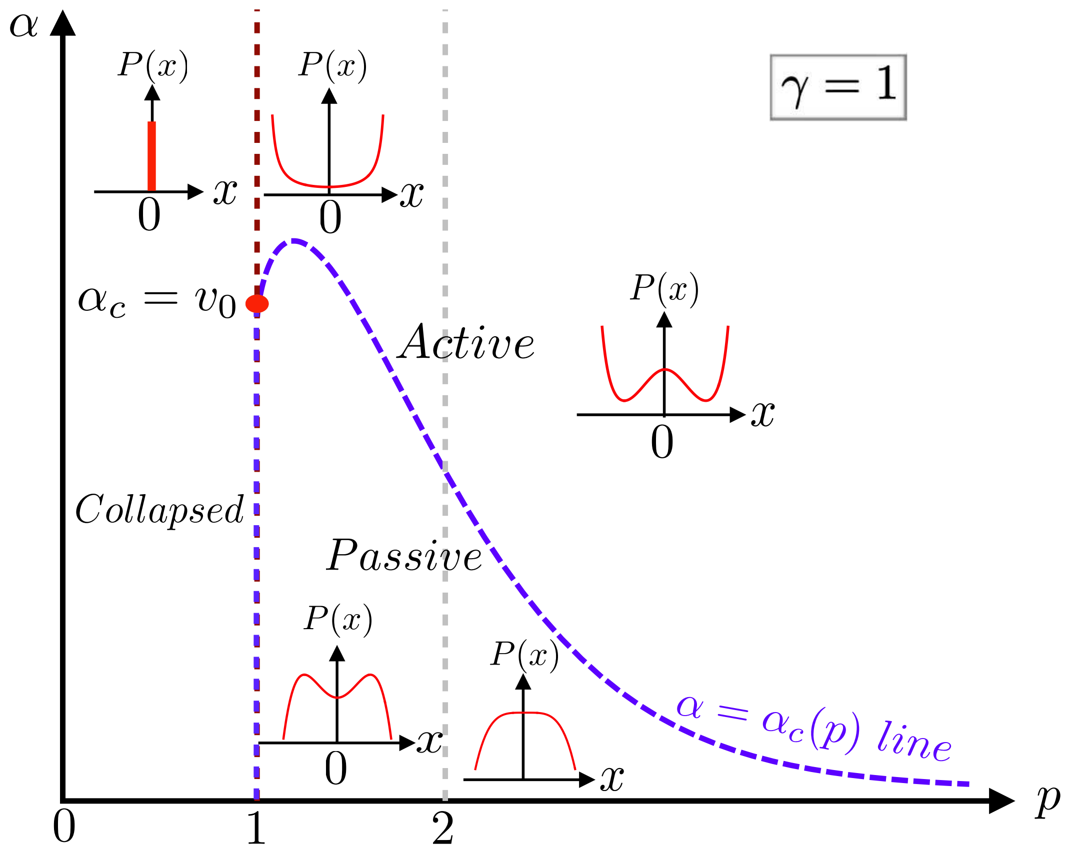
Transition from active to passive behavior

$$\frac{dX}{dt} = -V'(x) + v_0\,\sigma(t)$$

$$V(x) = \alpha\,|x|^p$$



- Phase diagram in the (p, α) plane — for fixed γ



A. Dhar, A. Kundu, S. N. Majumdar, S. Sabhapandit, G. S., PRE' 2019

Outline

- Two states RTP: stationary state in a confining potential $V(x)$

- Two particles ($N = 2$) with attractive interaction

- Many RTP's in interaction: the active Dyson Brownian motion

- Conclusion

Two interacting RTP's on the line

- Two RTP's $x_1(t), x_2(t)$ interacting via a potential $V(x_1 - x_2)$

$$\frac{dx_1(t)}{dt} = f(x_1 - x_2) + v_0\, \sigma_1(t)$$

$$\frac{dx_2(t)}{dt} = f(x_2 - x_1) + v_0\, \sigma_2(t)$$

two independent telegraphic noises $\mathrm{Exp}(\gamma)$

with $f(x) = -V'(x)$ and $V(x) = V(-x)$

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→ focus on $y(t)$

RTP with 3 internal states

$$\frac{dy}{dt} = 2f(y) + v_0(\sigma_1(t) - \sigma_2(t)) \qquad f(y) = -V'(y)$$

``telegraphic" noise with THREE

states: $-2v_0, 0, +2v_0$

U. Basu, S. N. Majumdar, A. Rosso, S. Sabhapandit, G. S., J. Phys. A '20
P. Le Doussal, S. N. Majumdar, G. S., PRE '21

RTP with 3 internal states

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- If $f(y)$ is sufficiently confining, there is a bound state

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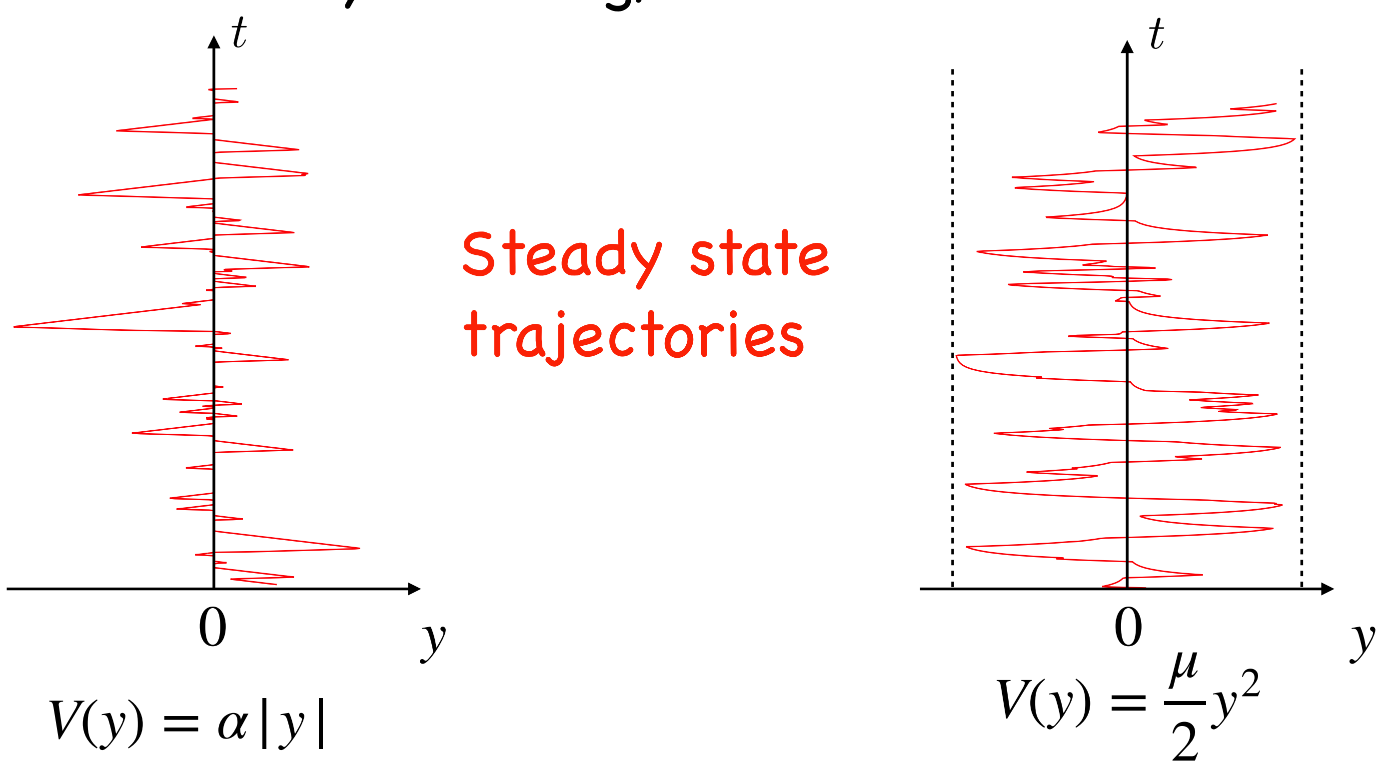

RTP with 3 internal states

$$\frac{dy}{dt} = 2f(y) + v_0(\sigma_1(t) - \sigma_2(t))$$

- If a stationary solution exists, the stationary PDF $P(y)$ satisfies

P. Le Doussal, S. N. Majumdar, G. S., PRE '21

$$f(y)\left(v_0^2 - f(y)^2\right) P''(y) + \left(\left(v_0^2 - 3f(y)^2\right)\left(\gamma + 2f'(y)\right) + \frac{f(y)\left(f(y) - v_0\right)\left(f(y) + v_0\right) f''(y)}{2\gamma + f'(y)} \right) P'(y)$$

$$+ \left(\frac{\gamma\left(v_0^2 - 3f(y)^2\right) f''(y)}{2\gamma + f'(y)} - f(y)\left(\gamma + 2f'(y)\right)\left(2\gamma + 3f'(y)\right) \right) P(y) = 0$$

+ boundary conditions

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P. Le Doussal, S. N. Majumdar, G. S., PRE '21

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\+ boundary conditions

- We found exact solutions for
  - $V(y) = \bar{c}\,|\,y\,| \quad , \quad \bar{c} > 0$
  - $V(y) = \frac{\mu}{2} y^2 \quad , \quad \mu > 0$

RTP with 3 internal states

- Exact solution for $V(y) = \alpha |y|$, i.e., $f(y) = -V'(y) = -\alpha \, \text{sign}(y)$

$$\frac{dy}{dt} = -2\alpha \, \text{sign}(y) + v_0(\sigma_1(t) - \sigma_2(t))$$

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$$P(y) = \frac{\bar{c}^2}{v_0^2 + \bar{c}^2}\delta(y) + \frac{\bar{c}\gamma v_0^2}{v_0^4 - \bar{c}^4} \, e^{-\frac{2\gamma\bar{c}}{v_0^2 - \bar{c}^2}|y|}$$

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bound state

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bound state

see also A. B. Slowman, M. R. Evans, R. Blythe, PRL '16 & JPA '17

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bound state

see also A. B. Slowman, M. R. Evans, R. Blythe, PRL '16 & JPA '17

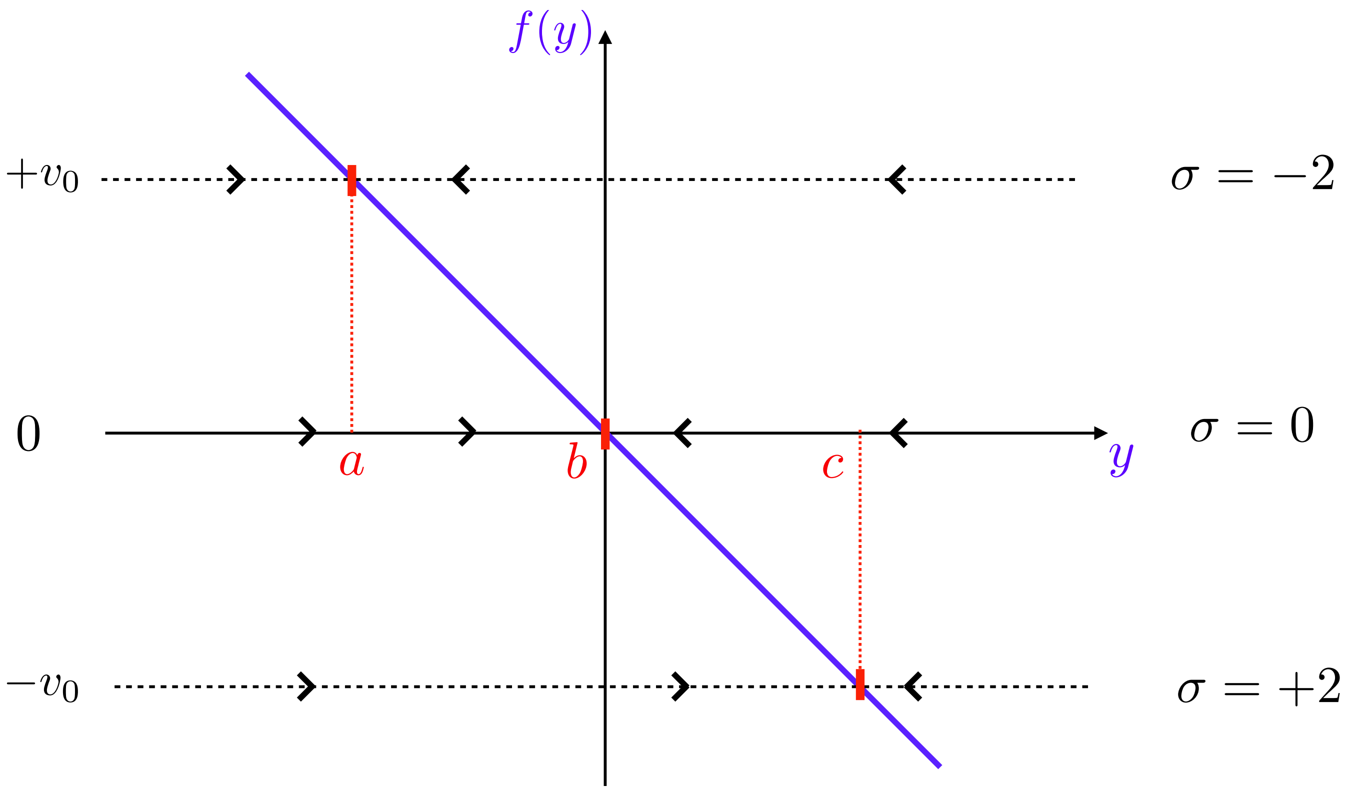
→ recent extension to $N \gg 1$ particles

L. Touzo, P. Le Doussal, arXiv:2308.06118

RTP with 3 internal states

- Exact solution for $V(y) = \dfrac{\mu}{2} y^2$, i.e., $f(y) = -V'(y) = -\mu\, y$

$$\frac{dy}{dt} = -2\mu\, y + v_0(\sigma_1(t) - \sigma_2(t))$$



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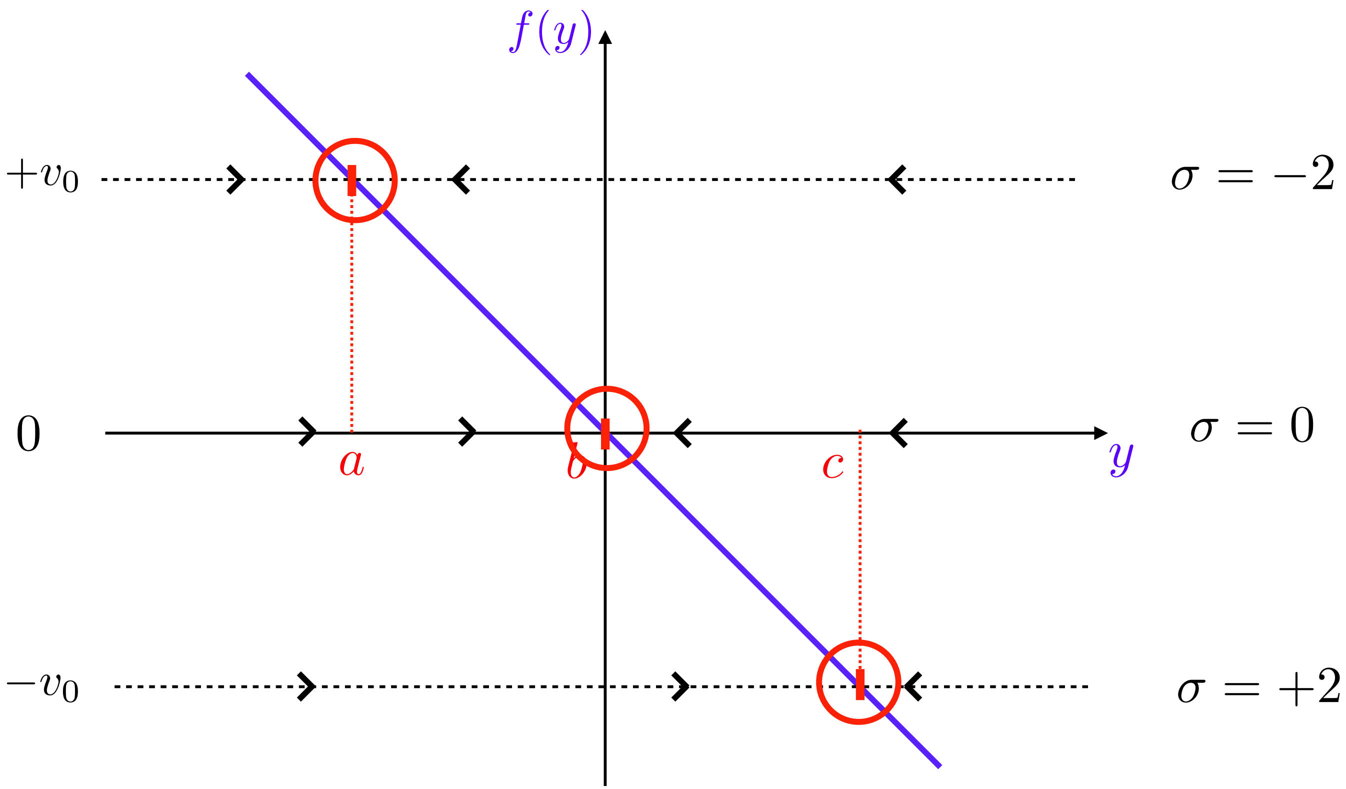

one expects singularities near the three fixed points $x_-, 0, x_+$

RTP with 3 internal states

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$$\frac{dy}{dt} = -2\mu\, y + v_0(\sigma_1(t) - \sigma_2(t))$$

$$P(y) = A_1 \left[{}_2F_1\left(1-\frac{\beta}{2}, \frac{3}{2}-\beta, \frac{3-\beta}{2}; \left(\frac{\mu y}{v_0}\right)^2\right) + \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{3-\beta}{2})\Gamma(\beta+\frac{1}{2})}{(1-2\beta)\Gamma(\frac{\beta+1}{2})} \left(\frac{\mu y}{v_0}\right)^{\beta-1} {}_2F_1\left(\frac{1}{2}, 1-\frac{\beta}{2}, \frac{\beta+1}{2}; \left(\frac{\mu x}{v_0}\right)^2\right)\right] \quad , \quad -\frac{v_0}{\mu} \le y \le \frac{v_0}{\mu}$$

with $\beta = \frac{\gamma}{\mu}$

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RTP with 3 internal states

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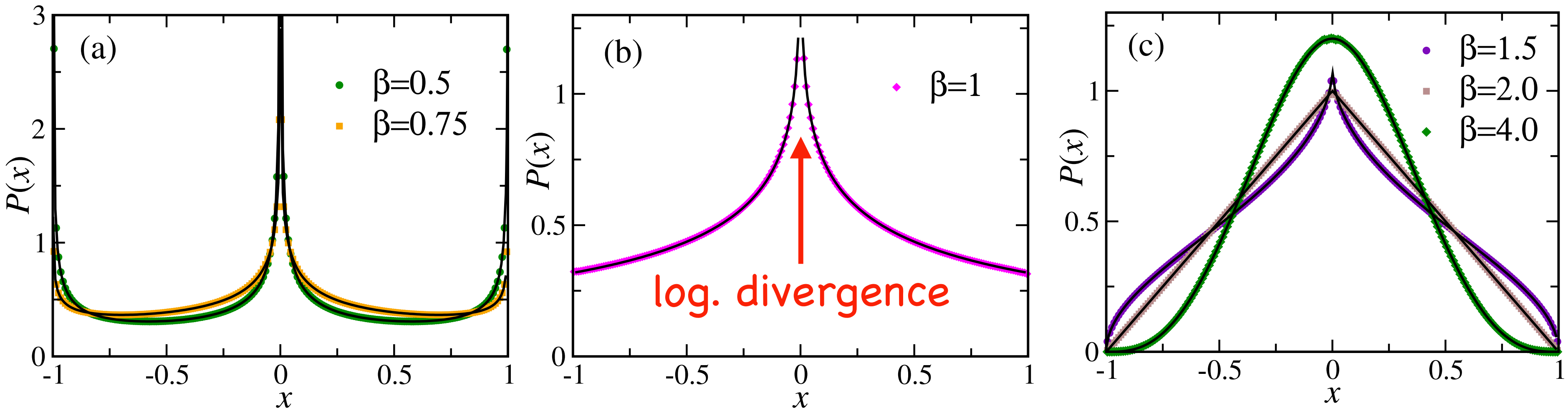
$$\frac{dy}{dt} = -2\mu y + v_0(\sigma_1(t) - \sigma_2(t))$$

- Shape transitions as $\beta = \gamma/\mu$ is varied

« active » « passive »



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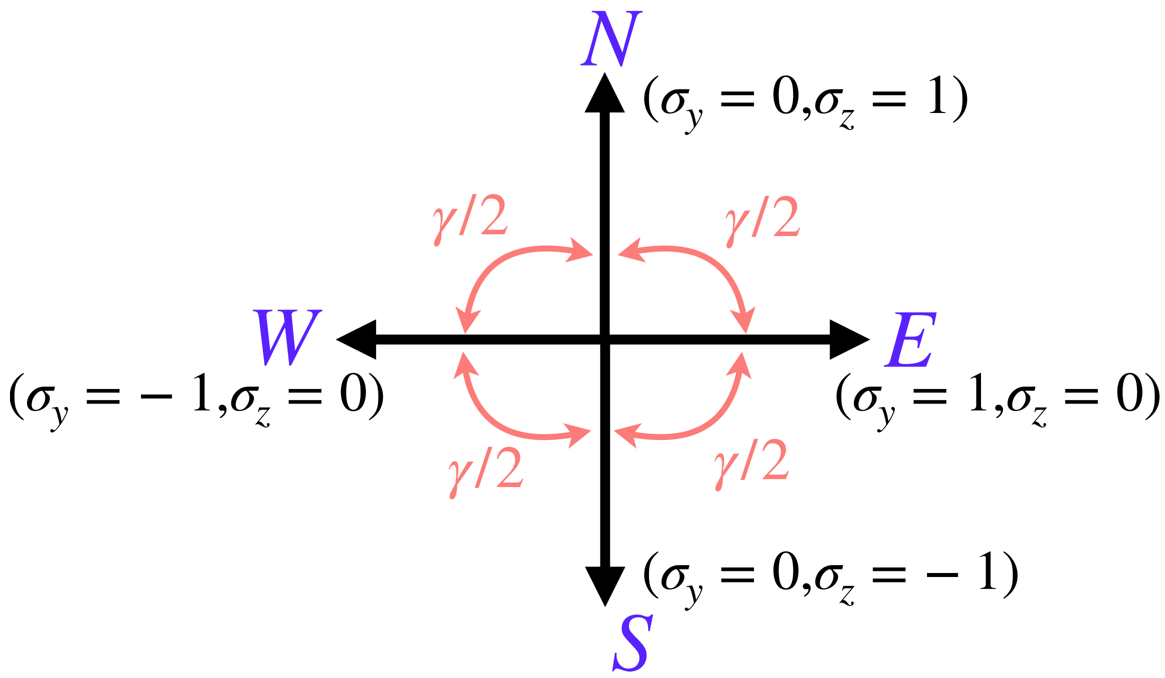
RTP with 3 internal states and a 2d model

- Relation to a 2d RTP $(y(t), z(t))$ in a harmonic potential $V(y,z) = \frac{\mu}{2}(y^2 + z^2)$

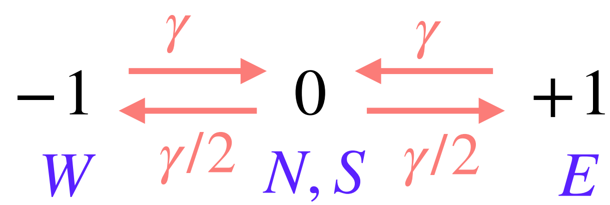
$$\dot{y}(t) = -\mu\, y(t) + v_0\, \sigma_y(t)$$

$$\dot{z}(t) = -\mu\, z(t) + v_0\, \sigma_z(t)$$

where the noise $\vec{\sigma}(t) = (\sigma_y(t), \sigma_z(t))$ has 4 internal states:



$\sigma_y(t)$: 3-state process



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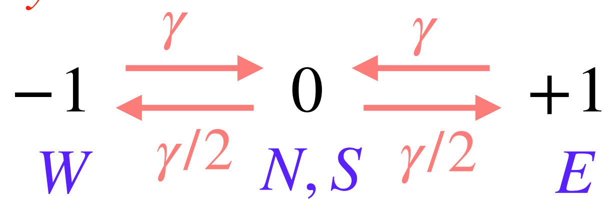

the $y-$component of the process coincides with the interparticle distance of $N = 2$ interacting particles

$$\frac{dy}{dt} = -2\mu\, y + v_0(\sigma_1(t) - \sigma_2(t))$$

``telegraphic'' noise with THREE states: $-2v_0, 0, +2v_0$

U. Basu et al. J. Phys. A '20
P. Le Doussal, S. N. Majumdar, G. S., PRE '21

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The active Dyson Brownian Motion

L. Touzo, P. Le Doussal, G. S., EPL `22

- Consider N particles on the line $x_1(t), x_2(t), \cdots, x_N(t)$ evolving via

$$\dot{x}_i(t) = -\mu x_i(t) + \frac{2g}{N} \sum_{j \neq i} \frac{1}{x_i(t) - x_j(t)} + v_0\, \sigma_i(t) + \sqrt{\frac{2T}{N}} \xi_i(t)$$

``telegraphic'' two-state noises

Gaussian white noises

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``telegraphic" two-state noises ($\sigma_i(t)$)

Gaussian white noises ($\xi_i(t)$)

- For $g = 0$ & $T = 0$, this is the noninteracting model studied before

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L. Touzo, P. Le Doussal, G. S., EPL '22

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``telegraphic'' two-state noises (the $\sigma_i(t)$ term)

Gaussian white noises (the $\xi_i(t)$ term)

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- For $v_0=0$, this is the well known Dyson Brownian Motion

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L. Touzo, P. Le Doussal, G. S., EPL '22

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→ the stationary state is related to the Gaussian β-ensemble of random matrix theory, $\beta = 2g/T$

The active Dyson Brownian Motion

L. Touzo, P. Le Doussal, G. S., EPL `22

- Consider N particles on the line $x_1(t), x_2(t), \cdots, x_N(t)$ evolving via

$$\dot{x}_i(t) = -\mu x_i(t) + \frac{2g}{N} \sum_{j \neq i} \frac{1}{x_i(t) - x_j(t)} + \underbrace{v_0\, \sigma_i(t)}_{\text{``telegraphic'' two-state noises}} + \sqrt{\frac{2T}{N}}\underbrace{\xi_i(t)}_{\text{Gaussian white noises}}$$

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→ in particular, the stationary average density for large N converges to the Wigner semi-circle

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``telegraphic" two-state noises (for $\sigma_i(t)$)

Gaussian white noises (for $\xi_i(t)$)

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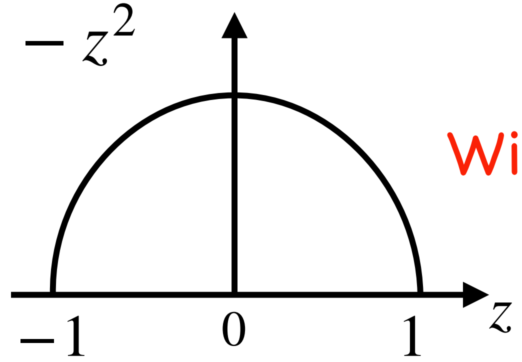
$$\rho(x,t) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i(t)) \xrightarrow[t,\, N\to\infty]{} \frac{1}{x_e}\rho_{\rm sc}\left(\frac{x}{x_e}\right) \quad , \quad x_e = 2\sqrt{\frac{g}{\mu}}$$

where $\quad \rho_{\rm sc}(z) = \frac{2}{\pi}\sqrt{1-z^2}$



Wigner semi-circle

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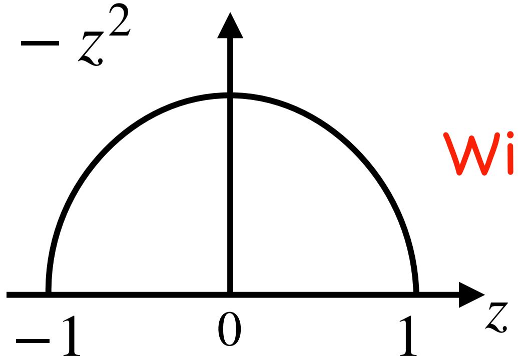

Wigner semi-circle

Q: what happens for $v_0 > 0$?

The active Dyson Brownian Motion at finite N

L. Touzo et al. EPL '22

- Focus on a purely active noise

$$\dot{x}_i(t) = -\mu\, x_i(t) + \frac{2\,g}{N} \sum_{j \neq i} \frac{1}{x_i(t) - x_j(t)} + v_0\, \sigma_i(t)$$

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- Two dimensionless parameters

$$\frac{v_0^2}{\mu\, g} \quad \& \quad \frac{\gamma}{\mu}$$

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- The noise term is finite ⟶ Particles can not cross !

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$$\Rightarrow \quad \lim_{N\to\infty} x_{+,N} = x_\infty = v_0 + 2\sqrt{g}$$

- The actual support for $N\to\infty$ turns out to be strictly smaller than $[-x_\infty, +x_\infty]$ (when $g = O(1)$)

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L. Touzo et al. EPL '22

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- Large number of fixed points $\vec{x}^\star$ (typically 2^N), all of them are stable

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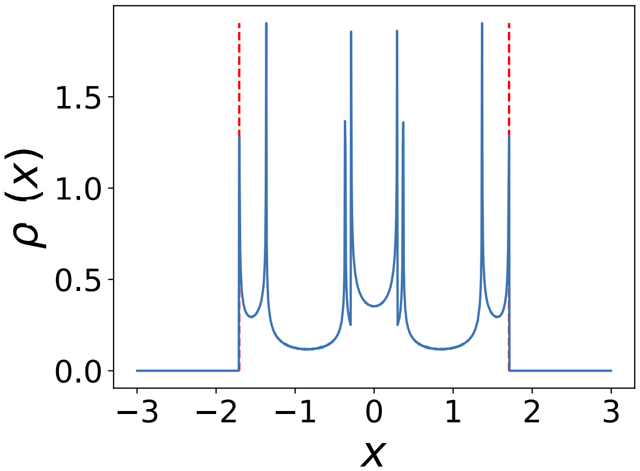
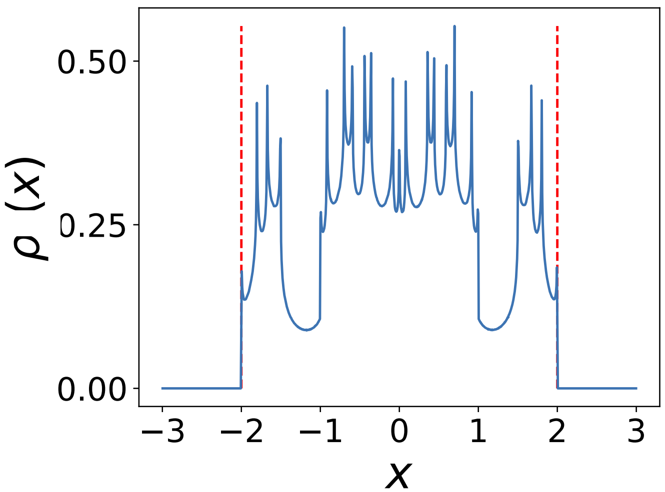
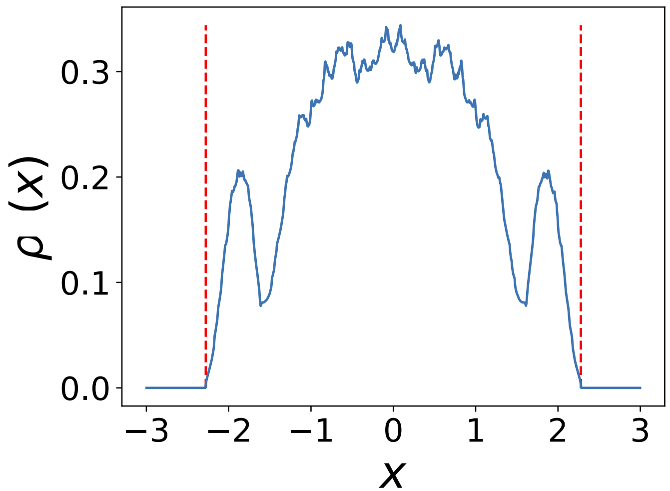


$N = 2$



$N = 3$



$N = 5$

The active Dyso` Brow`ia` Motio` as $N \to \infty$

L. Touzo et al. EPL `22

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- Particles ca` `ot cross a`d `ot excha`geable !

→ preve`ts us to apply the sta`dard hydro. approach à la Dea`-Kawasaki (DK)

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L. Touzo et al. EPL '22

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- Particles can not cross and not exchangeable !

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Rk: a variant of the model where particles with opposite 'spins' $\sigma \neq \sigma'$ can cross is well described by DK

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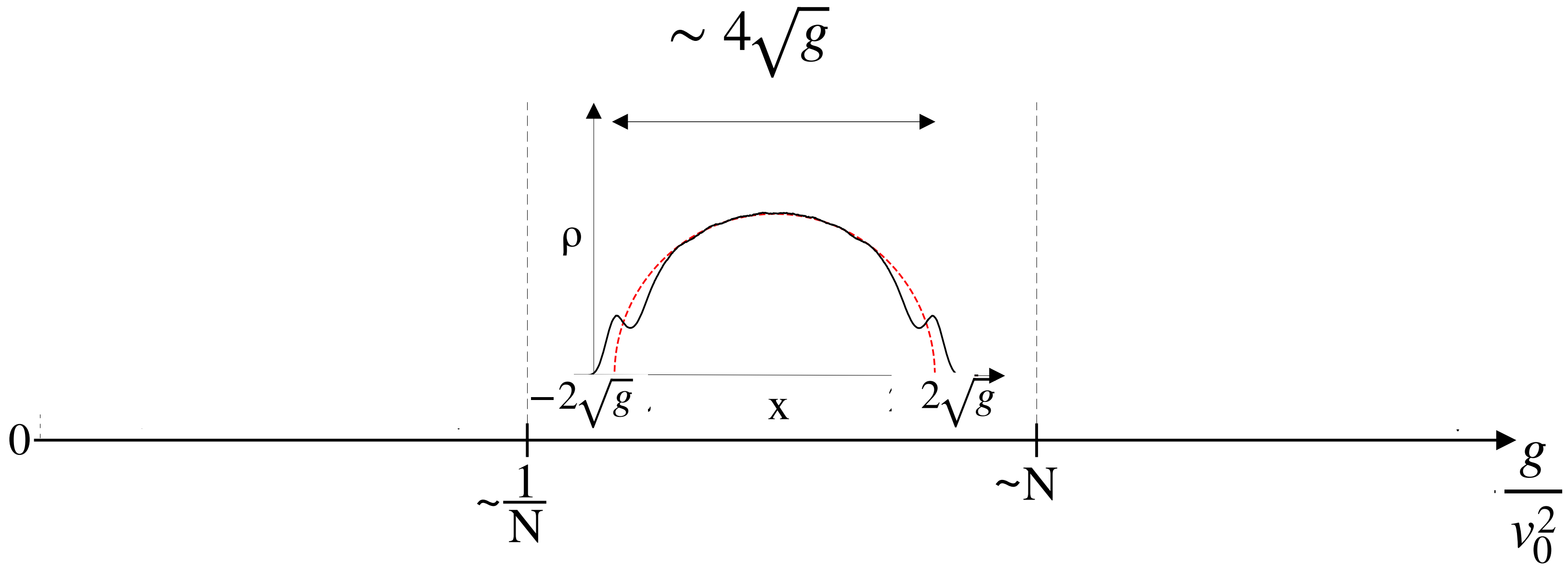

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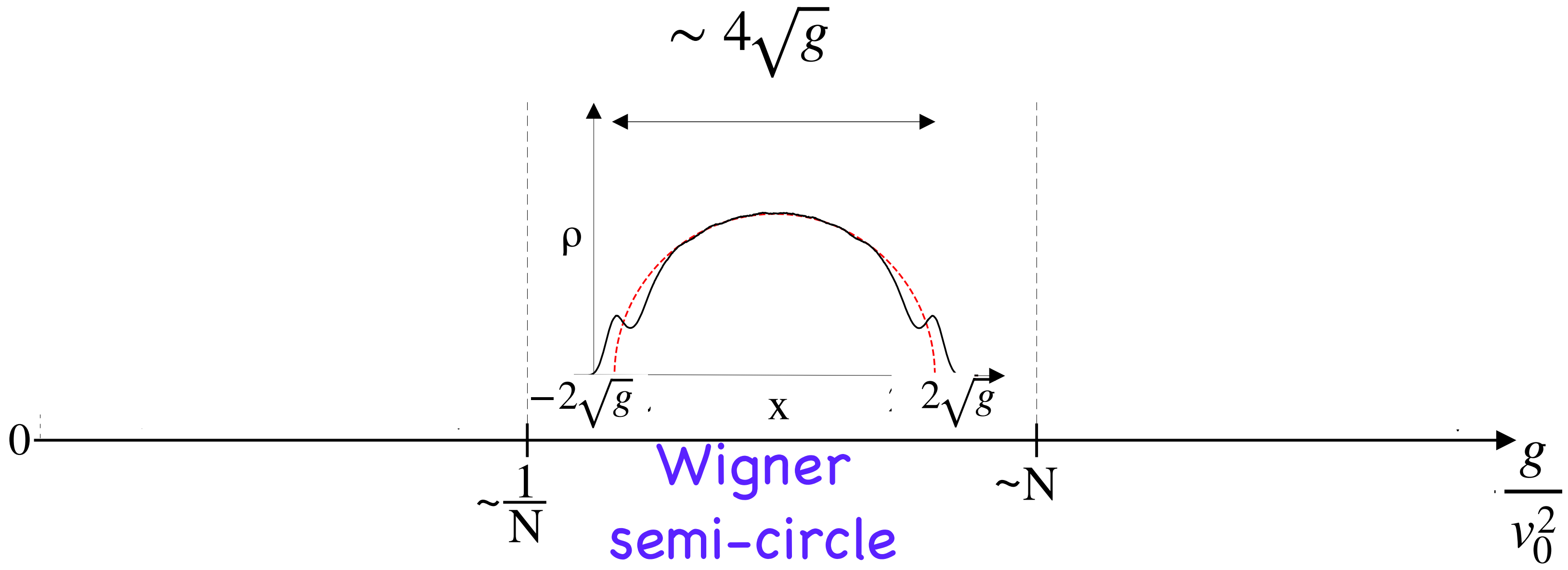

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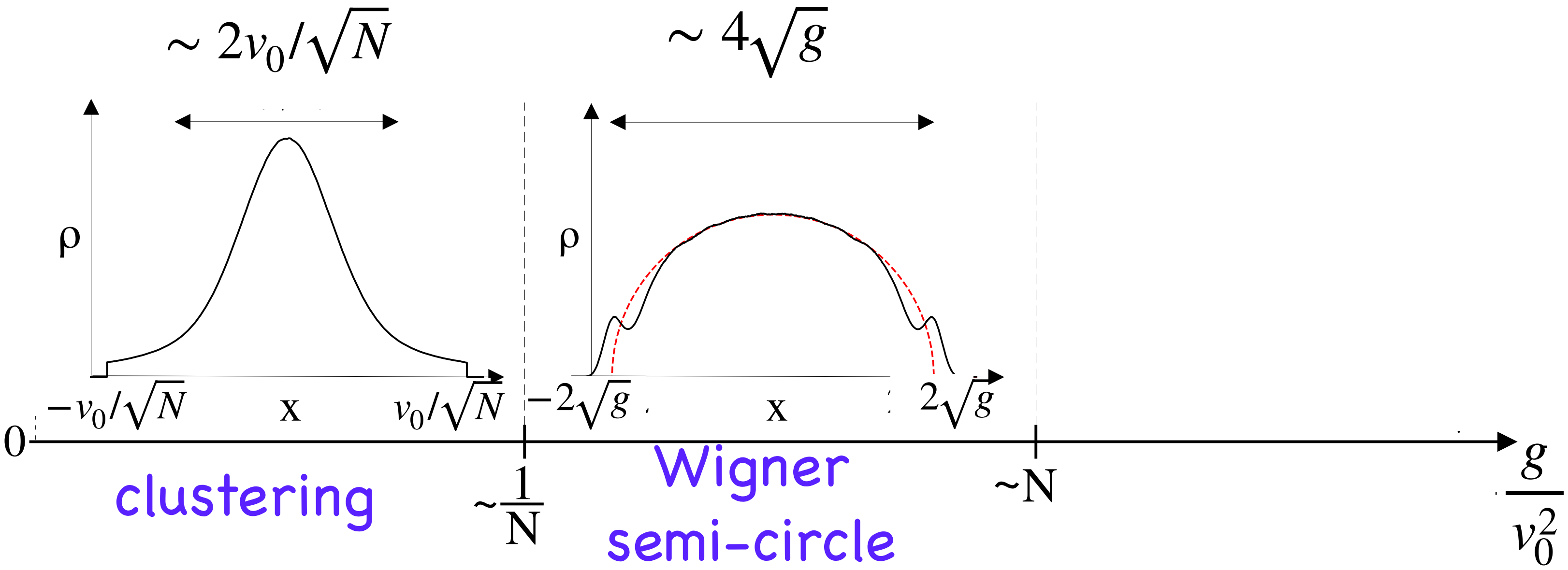

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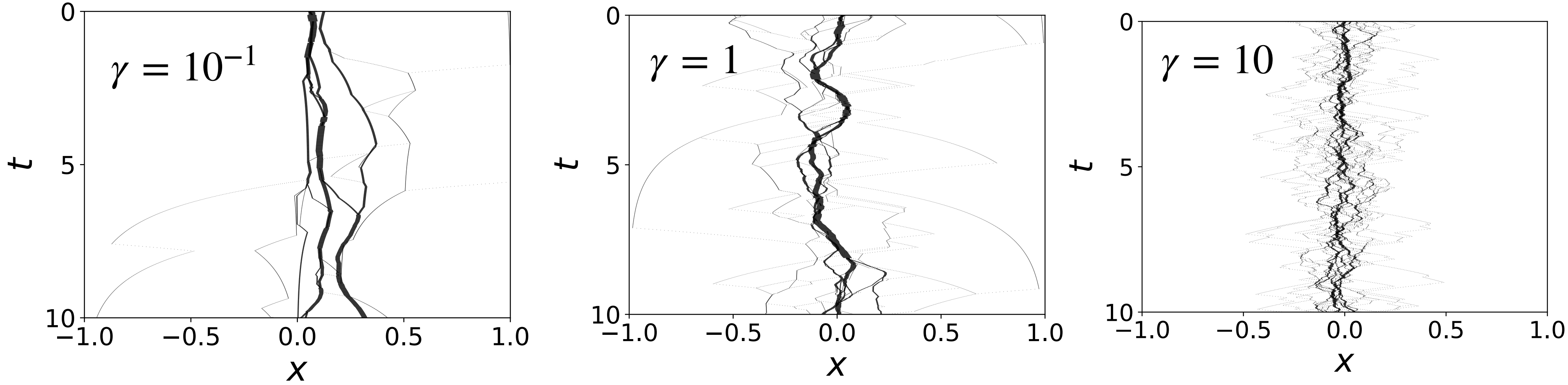

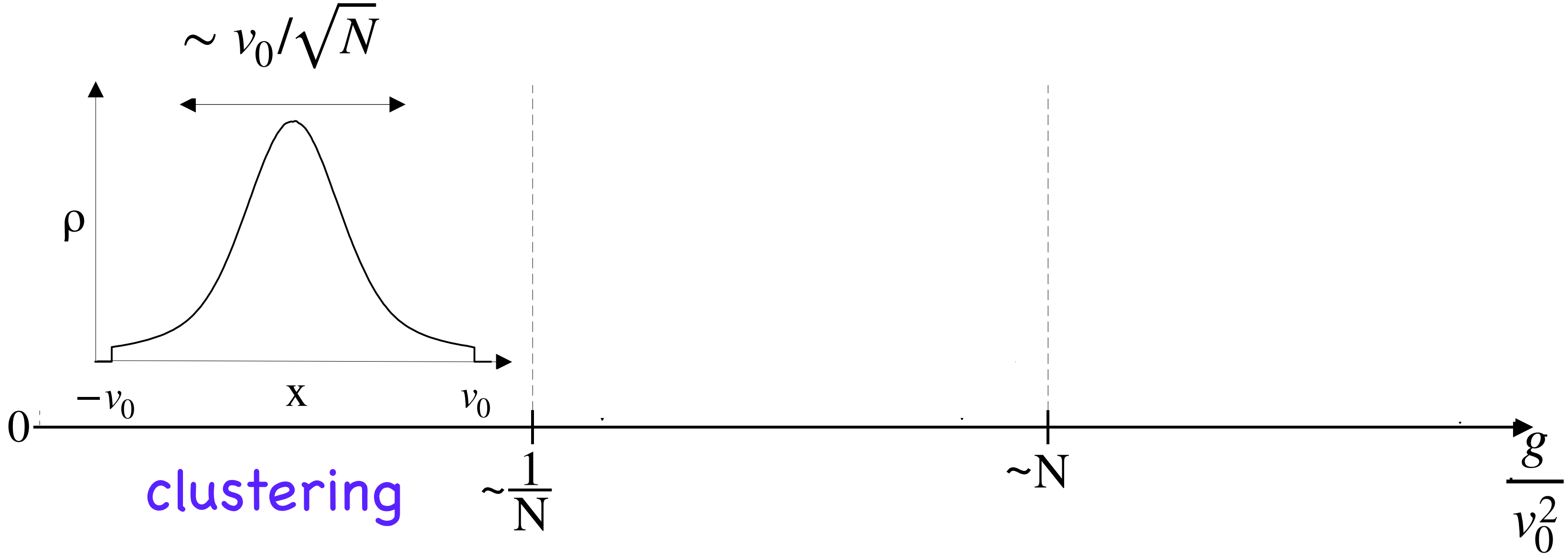

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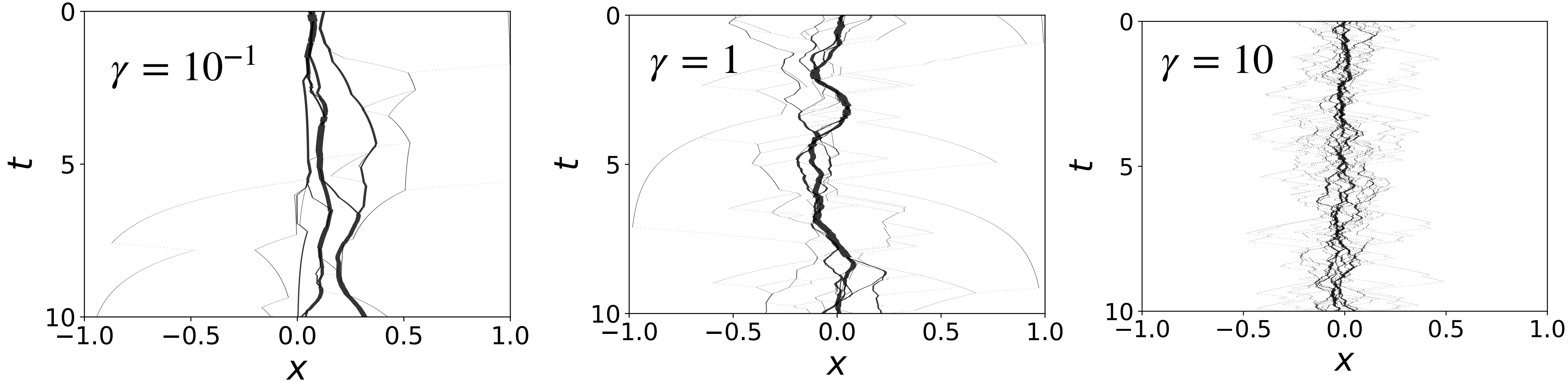

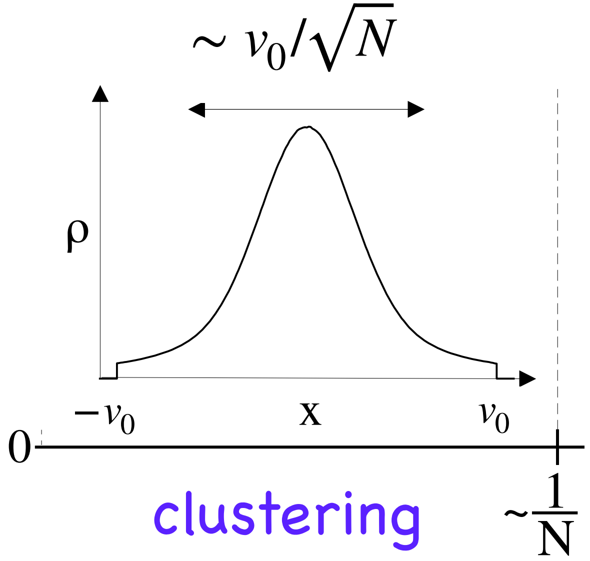

$$\rho(x) \sim \frac{\sqrt{N}}{v_0}\,\phi\left(\sqrt{N}\frac{x}{v_0}\right)$$

$$\phi(z) \propto \frac{1}{z^3} \quad , \quad z \to \infty$$

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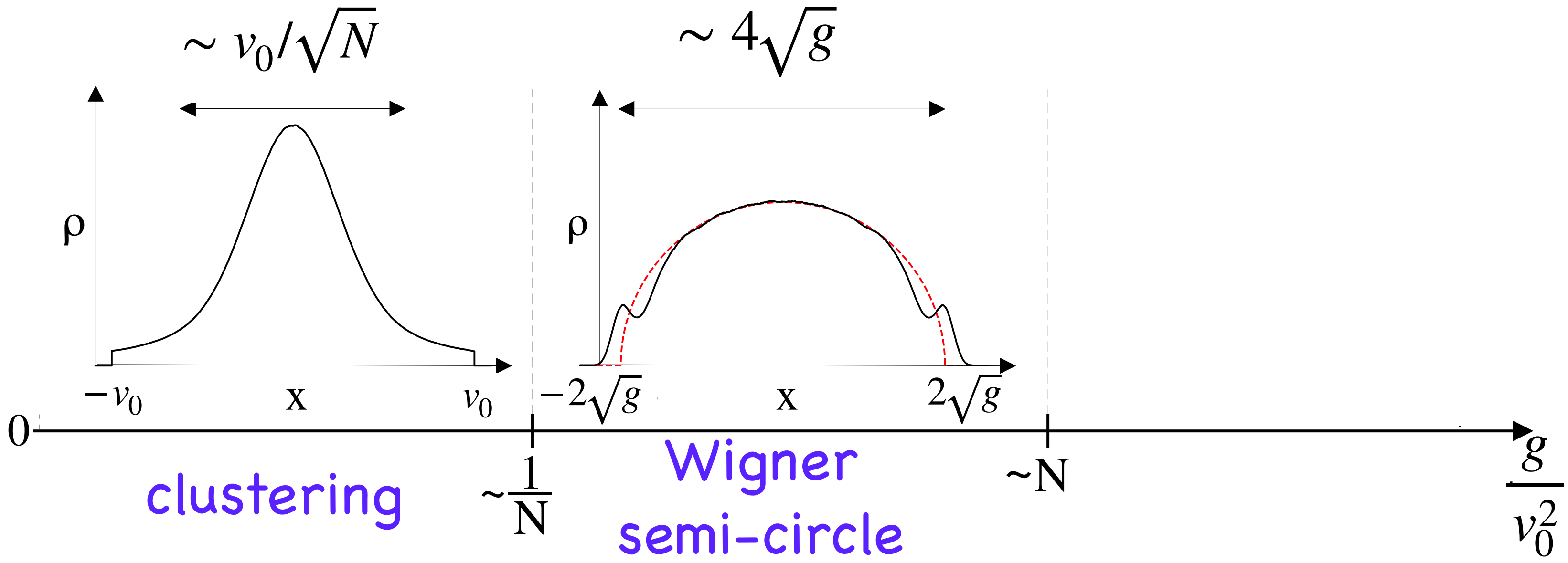

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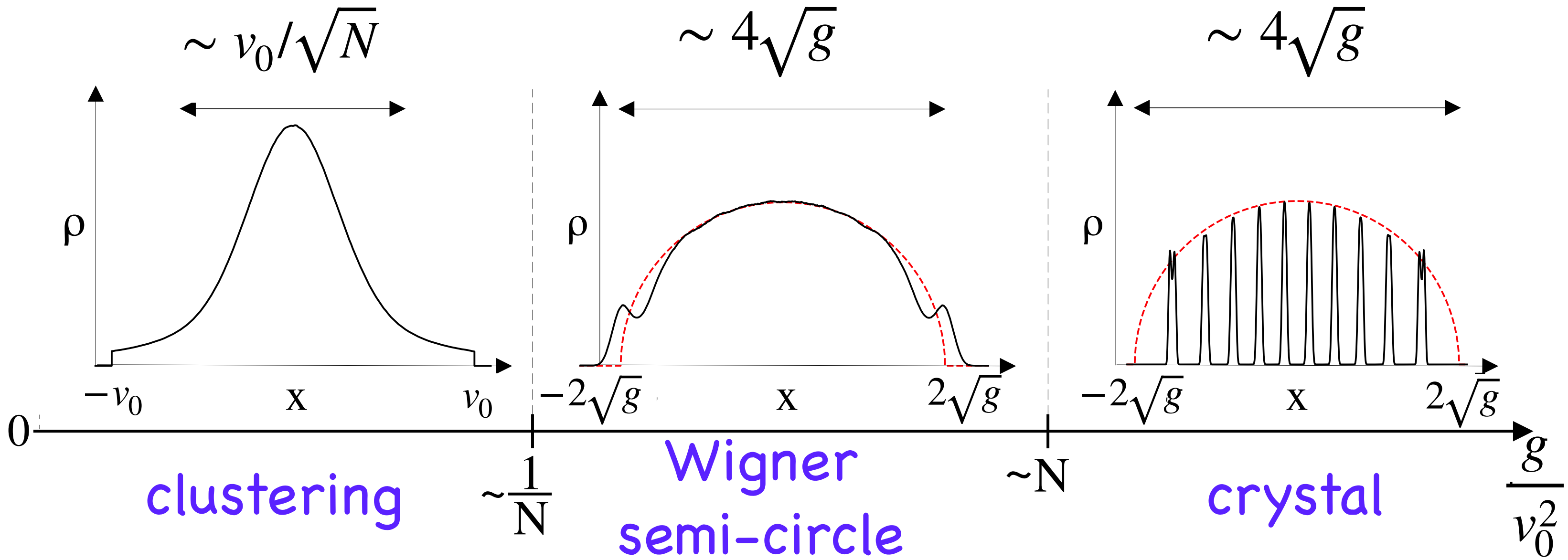

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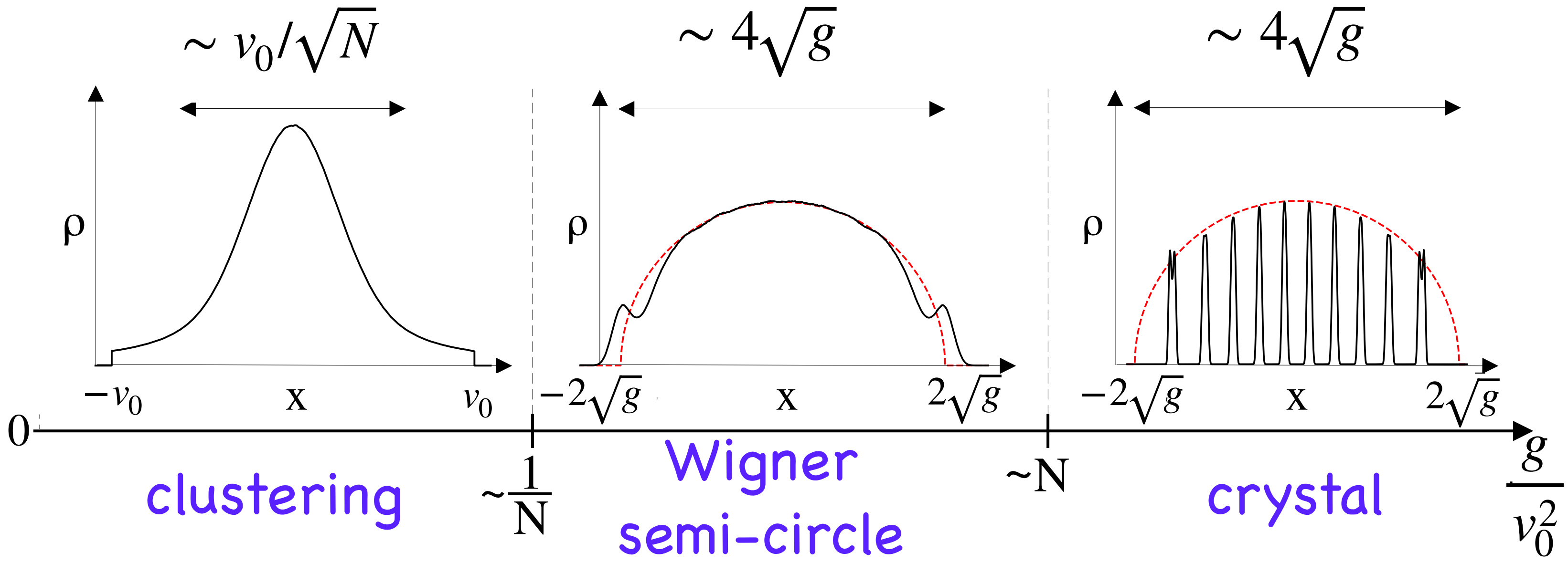
- The crystal limit $g/v_0^2 \to \infty$

$$x_i \longrightarrow \sqrt{\frac{2g}{N}}\, y_i$$

zeros of Hermite polynomial $H_N(y_i) = 0$



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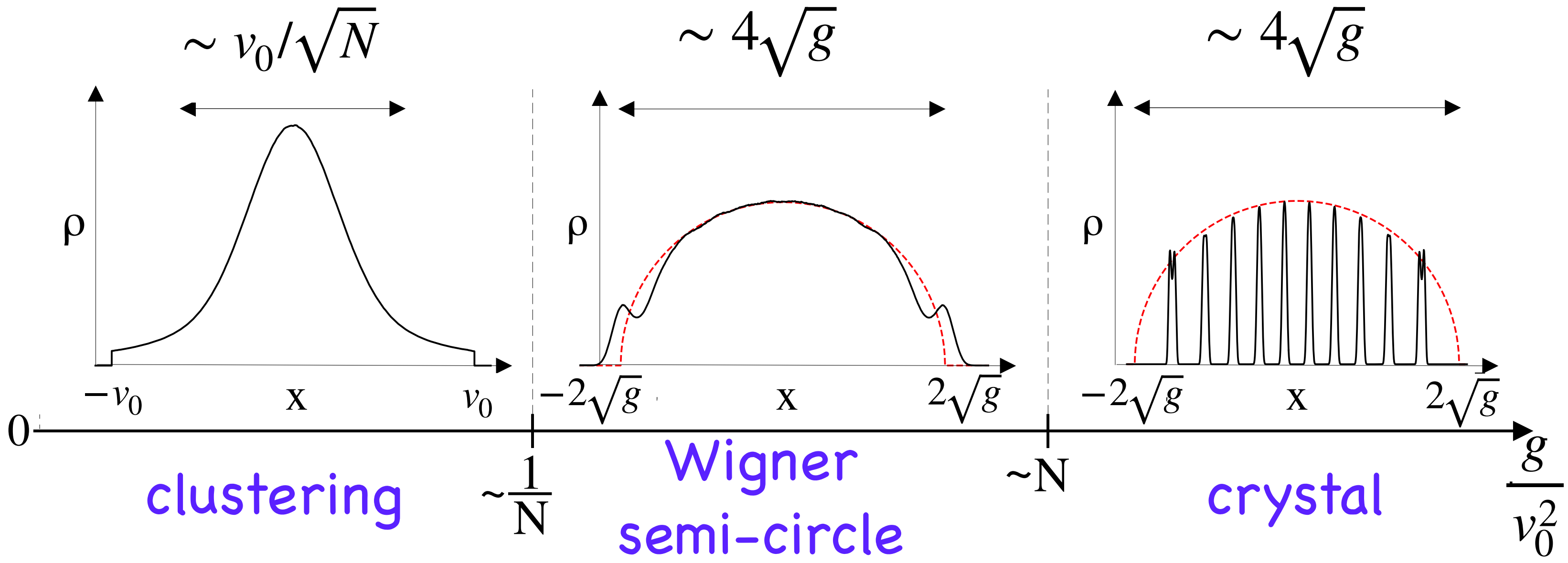
⇒ small v_0 expansion



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- Small v_0 expansion: « active phonons »

L. Touzo, P. Le Doussal, G. S. arXiv:2302.02937

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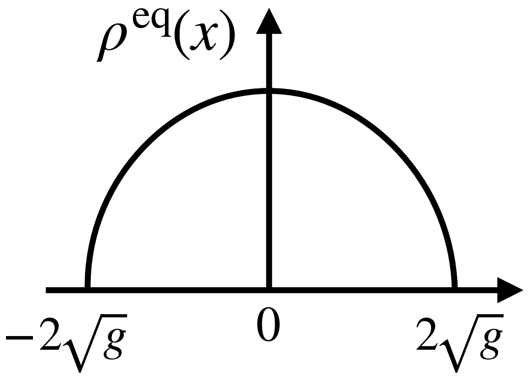

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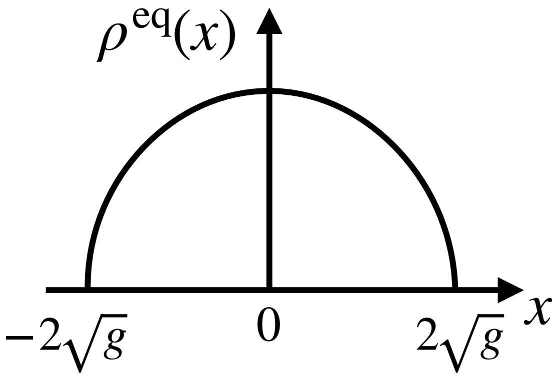
Wigner semi-circle



The active Dyson Brownian Motion as $N \to \infty$

$$\dot{x}_i(t) = -x_i(t) + \frac{2g}{N} \sum_{j \neq i} \frac{1}{x_i(t) - x_j(t)} + v_0\, \sigma_i(t)$$

- Small v_0 expansion: « active phonons »

L. Touzo, P. Le Doussal, G. S.
arXiv:2302.02937

$$\delta x_i = x_i - x_i^{\rm eq} \quad \text{where} \quad x_i^{\rm eq} = \sqrt{\frac{2g}{N}}\, y_i$$

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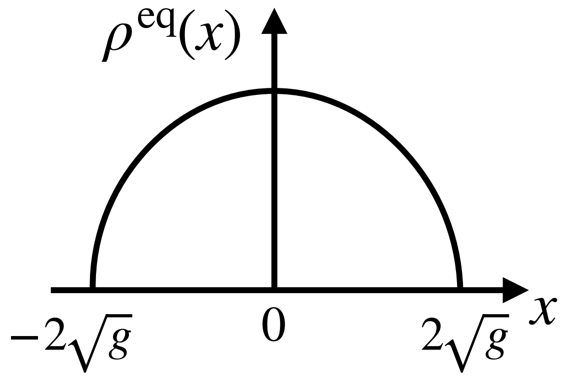

- For small v_0 and in the limit $\gamma \to 0^+$ one finds (using Hessian)

$$\langle \delta x_i \delta x_j \rangle = v_0^2 \sum_{k=1}^{N} \frac{1}{k^2} \frac{u_k(y_i) u_k(y_j)}{\sum_{l=1}^{N} u_k(y_l)^2} + O(v_0^3) \quad \text{with} \quad u_k(y) = \frac{H_N^{(k)}(y)}{H_N'(y)}$$

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- Large N analysis

  - In the bulk   L. Touzo, P. Le Doussal, G. S., arXiv:2302.02937

$$\langle \delta x_i \delta x_j \rangle \simeq \frac{v_0^2}{N} \mathscr{C}_b \left(\frac{y_i}{\sqrt{2N}}, \frac{y_j}{\sqrt{2N}} \right)$$

$$\mathscr{C}_b(x, y) = \frac{\pi \arccos(\max(x, y)) - \arccos(x) \, \arccos(y)}{2\sqrt{1 - x^2}\sqrt{1 - y^2}}$$

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⟹ this indicates that the Wigner semi-circle holds if $v_0^2/N \ll g$

The active Dyson Brownian Motion as $N \to \infty$

L. Touzo et al. EPL '22

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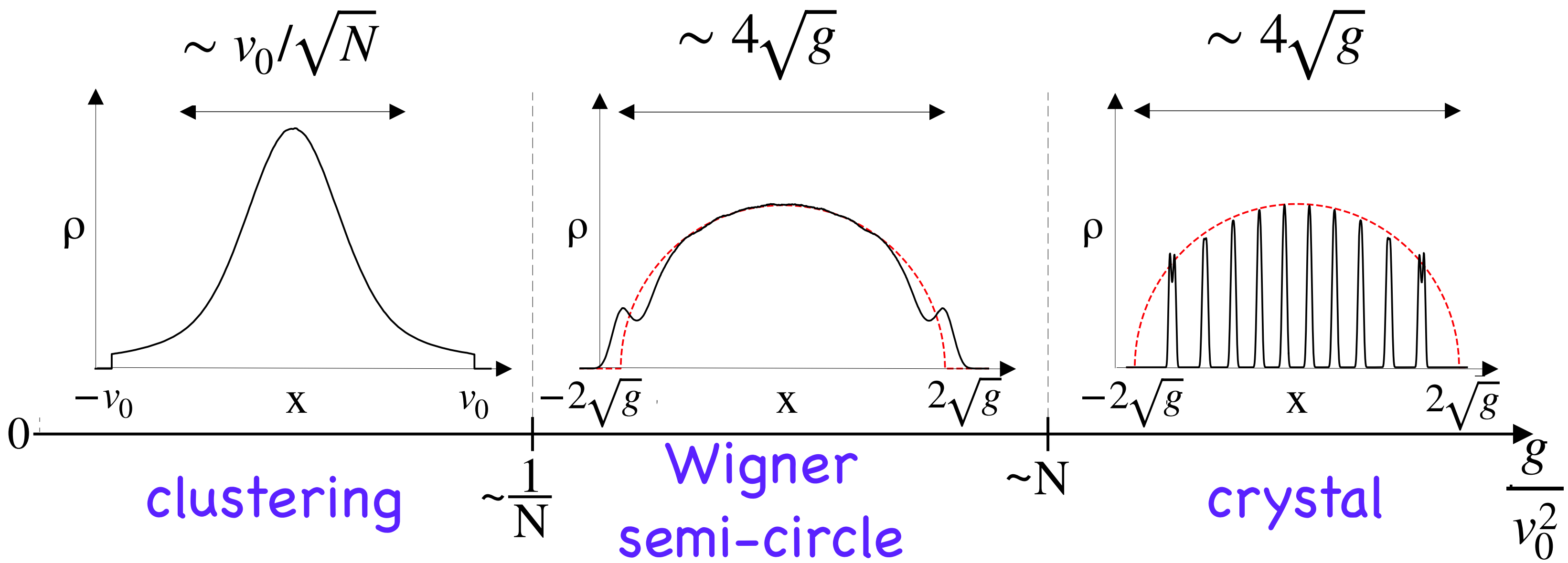

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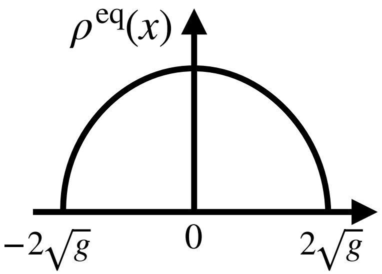
- Large N analysis

  - At the edge



zeros of the Airy function $\mathrm{Ai}(a_i) = 0$

$$\langle \delta x_i\,\delta x_j\rangle \simeq \frac{v_0^2}{N^{2/3}}\mathscr{C}_e(a_i, a_j)$$

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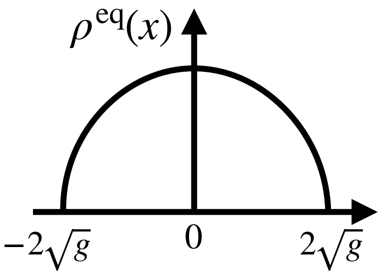

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Rk: a similar result was recently found for the Dyson Brownian motion in the limit $\beta \to \infty$ by Gorin & Kleptsyn (2009), with $1/x^2 \to 1/x$

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- Better understanding of the validity of hydrodynamic approach à la Dean-Kawasaki for active systems ?

Passive vs active particles

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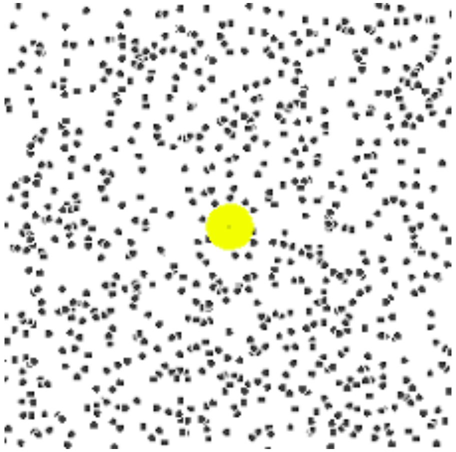
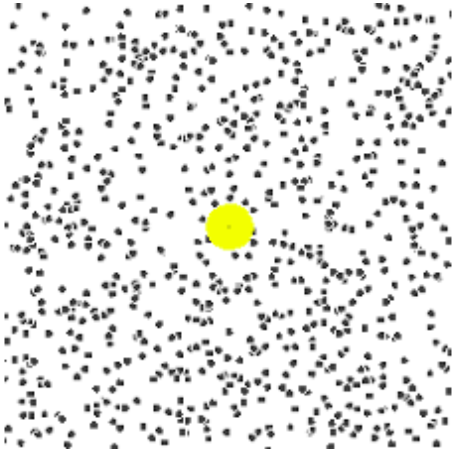
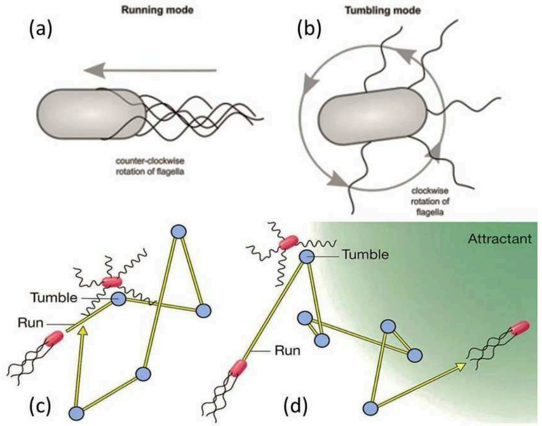


- Passive BM: random motion due to collisions with other molecules

Passive vs active particles





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- Active particle: the particle absorbs energy directly from the environment $\Longrightarrow$ a ballistic motion (Run) with a constant velocity $\vec{\mathbf{v}}$ during an exponentially distributed random time with mean γ^{-1} (persistence time), followed by a local reorientation of the velocity (Tumble)... another run...

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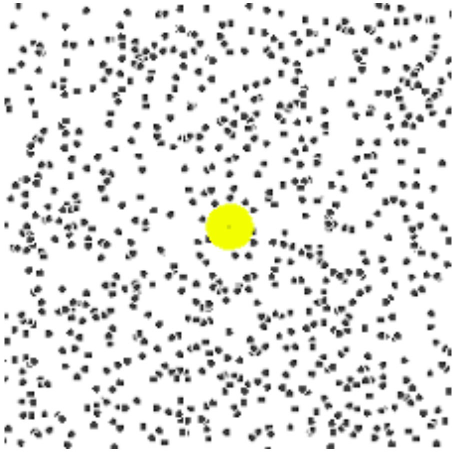

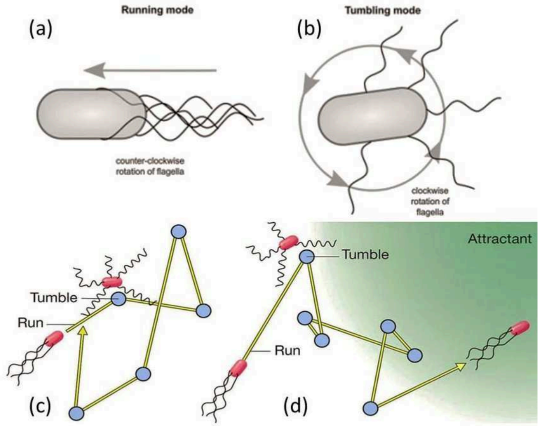

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Ex: widely used to model dynamics of living matter, like E. Coli

Berg (2004), Tailleur and Cates (2008), ...