RRI
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Hydrodynamic Scales of Integrable Many-Particle Systems

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Calogero fluid

arXiv: 2301.08504

integrable many-particle systems

Lieb-Liniger δ-Bose gas (1963)

XYZ spin chains

Toda lattice (quantum + classical)

→ Calogero fluid ← + + +

box-ball system

Korteweg-De Vries

nonlinear Schrödinger

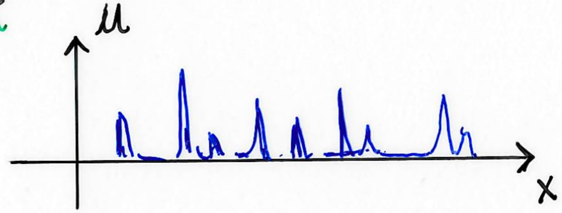
→ generalized Gibbs ensembles ← static

→ local GGEs ⇝ generalized hydrodynamics dynamics

// GGE averaged currents //

Euler hydrodynamics

- soliton gas

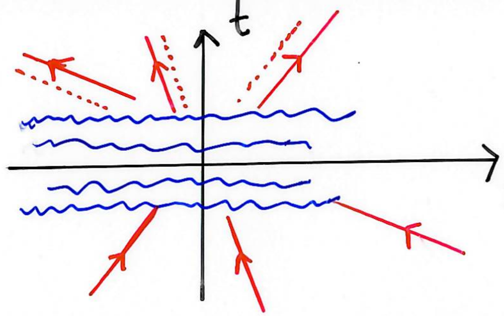


soliton-based G. El (2003)

- particle system quantum / classical

// particles are solitons //

scattering shift



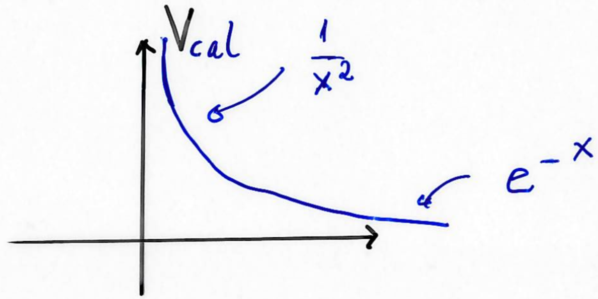
particle-based Doyon et al, De Nardis et al (2016)

→ Toda lattice $H = \sum_j \left(\frac{1}{2} p_j^2 + e^{-(q_{j+1} - q_j)} \right)$

$$\left.\begin{array}{l}\text{two-particle}\\ \text{two-soliton}\end{array}\right\} \text{scattering} \rightsquigarrow \text{2 distinct hydrodynamic scales}$$

1. Calogero fluid   classical q_j, p_j, $j = 1, \dots, N$ on $\mathbb{R}$ (1975)

$$H_N = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{i<j=1}^{N} V_{cal}(q_i - q_j)$$

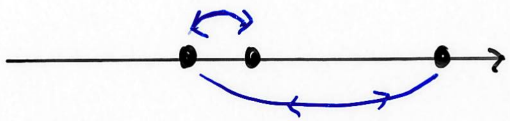


$$V_{cal}(x) = \frac{1}{\sinh^2(x)}$$

- no crossing $T_N = W_N \times \mathbb{R}^N$ Weyl $W_N = \{ q_1 < \dots < q_N \}$

$\Rightarrow$ high density $V_{cal}(x) = \frac{1}{x^2}$ rational   CMS   (ideal fluid)

$\Rightarrow$ low density



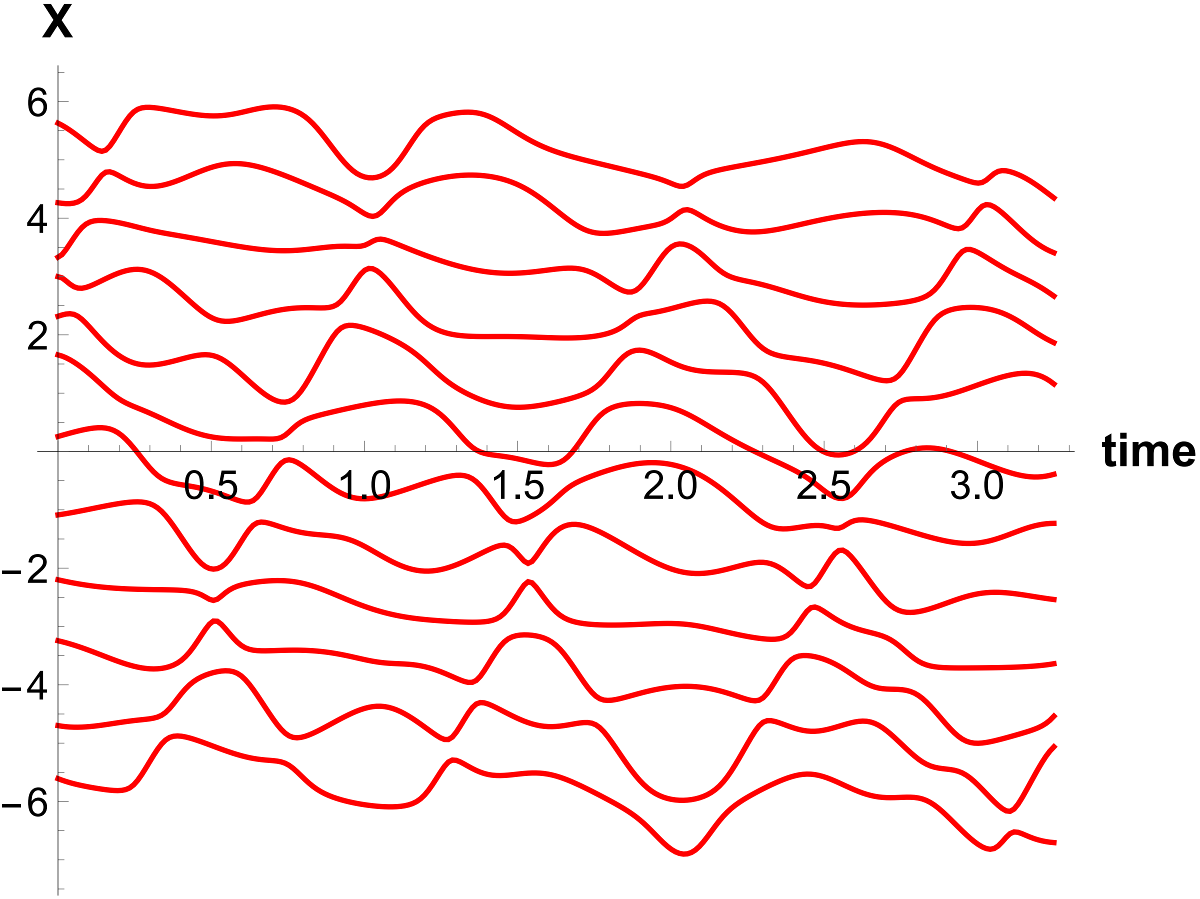
discard

$\Rightarrow$ $$H_{toda, N} = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} e^{-(q_{j+1} - q_j)}$$

stretch > 0 Toda (1967)

(free volume)

distinct < 0


x
6
4
2
−2
−4
−6
time
0.5
1.0
1.5
2.0
2.5
3.0

2. Lax matrix

$N \times N$

$$L_{ij} = \delta_{ij} p_j + i (1-\delta_{ij}) \frac{1}{\sinh(q_i - q_j)} \qquad L = L^*$$

Lax pair $L(q,p)$, $M(q,p)$ $\rightsquigarrow$ $\frac{d}{dt} L = [L, M]$

$\rightsquigarrow$ $L \psi_\alpha = \lambda_\alpha \psi_\alpha$ eigenvalues are conserved NON LOCAL $\{ (\sum_{j=1}^{N} p_j^2)^2$

- local fields density $\{$ total

$$Q^{[n]}(x) = \sum_{j=1}^{N} \delta(x - q_j) (L^n)_{jj} \quad \Big\{ \quad Q^{[n]} = \int dx\, Q^{[n]}(x) = \mathrm{tr}[L^n]$$

particle number $Q^{[0]}$, momentum $Q^{[1]}$, energy $Q^{[2]}$, ...

2nd Lax matrix $\frac{1}{\sinh} \Rightarrow \coth$ | construction of 1-, 2- solitons

|| uniqueness of Lax || | Kulkarni et al (2017)

3. GGE, generalized free energy

‖ wish list ‖

ℓ

ring

$$V_{cal,\ell}(x) = \sum_{m\in\mathbb{Z}} V_{cal}(x+m\ell)$$ double periodic Weierstrass

Lax pair ✓ ‖ more tricky ‖ $Q_\ell^{[n]} = \mathrm{tr}\,[(L_\ell)^n]$

⤳ GGE total charges $Q_\ell^{[n]}$

$$e^{-\sum_{n=0}^{\infty} \mu_n Q_\ell^{[n]}} = e^{-\mathrm{tr}[V(L_\ell)]}$$

relative to $d^N q\, d^N p$ on $\mathbb{T}_N$

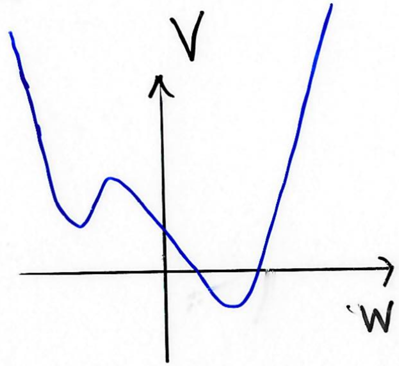
GGE parameters: $\frac{\ell}{N} = \nu$, V ‖ confining potential: $V(w) = \sum_{n=0}^{\infty} \mu_n w^n$ ‖

→ momentum space ←

- free energy

$$\lim_{\ell\to\infty} -\frac{1}{\nu}\frac{1}{N}\log Z_N(\nu, V) = F(\nu, V)$$

‖ functional of V ‖



‖ ideal gas

$$\prod_{j=1}^{N} e^{-V(p_j)}$$

4. Density of states

$L_{N,\ell}$ random matrix under GGE, DOS

eigenvalue

$$\rho_{Q,N}(w) = \frac{1}{N}\sum_{j=1}^{N} \delta(w-\lambda_j)$$

→ law of large numbers

$$\lim_{N\to\infty} \rho_{Q,N} = \rho_Q$$

← deterministic

$$\int_{\mathbb{R}} dw\, \rho_Q(w)\, w^n = \langle Q^{[n]}(0)\rangle_{GGE}$$

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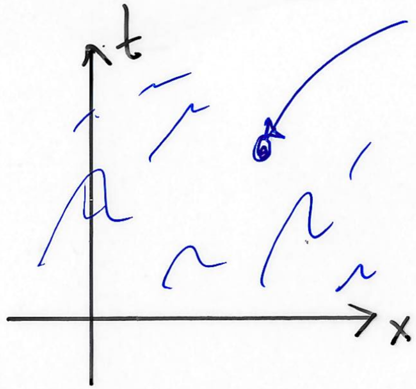
hydrodynamic picture, Lax filter



$\rho_Q(x,t;w)$ $*$

local DOS, ν

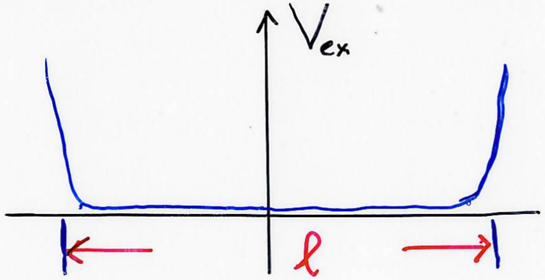
$*$ is governed by Euler equations ||

|| L filters slow degrees of freedom ||

5. External potential

$$T_N^{\blacktriangleright} = W_N \times \mathbb{R}^N$$

$$W_N = \{ x_1 < \dots < x_N \}$$



$$Z_N(\nu, V) = \int_{T_N^{\blacktriangleright}} d^N q \, d^N p \; e^{-\operatorname{tr} V(L)} \; e^{-\sum_{j=1}^{N} V_{ex}(q_j)}$$

Rujsenaars (1988 - 1995)

scattering coordinates

$$\lim_{t \to \infty} p_j(t) = \lambda_j \, , \quad \lim_{t \to \infty} q_j(t) - \lambda_j t = \phi_j \in \mathbb{R}$$

$\lambda \in W_N$ eigenvalues of L

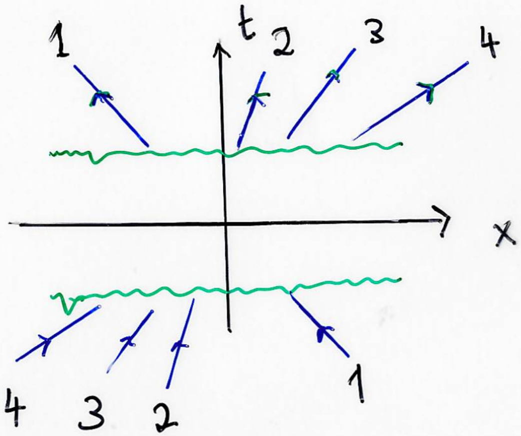
$\Phi : (\lambda, \phi) \mapsto (q, p)$ one-to-one, canonical

generic

Result (R 1988) algebraic construction of Φ

$$Z_N(\nu, V) = \int_{T_N^{\blacktriangleright}} d^N \lambda \, d^N \phi \; e^{-\sum_{j=1}^{N} V(\lambda_j)} \; e^{-\sum_{j=1}^{N} V_{ex}(q_j(\lambda, \phi))}$$

?



integrable

- special choice $V_{ex}(x) = e^{-\ell/2} \cosh x$

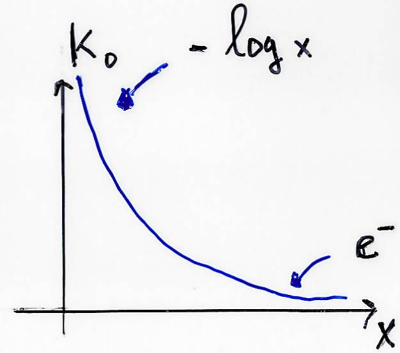
$$\sum_{j=1}^{N} V_{ex}(q_j) = \sum_{j=1}^{N} e^{-\ell/2} Y_j \cosh \phi_j$$

$$Y_j = \prod_{\substack{m=1 \\ m \neq j}}^{N} \left(1 + \frac{1}{(\lambda_m - \lambda_j)^2} \right)^{1/2}$$

confining!

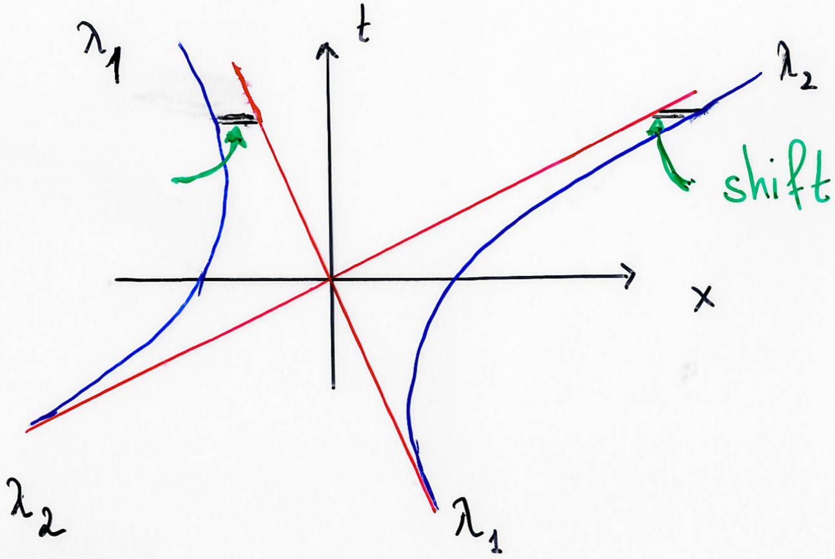
modified Bessel

$$\rightsquigarrow \quad Z_N(\nu, V) = \frac{1}{N!} \int_{\mathbb{R}^N} d^N\lambda \prod_{j=1}^{N} e^{-V(\lambda_j)} \prod_{j=1}^{N} 2K_0(2e^{-\ell/2} Y_j)$$



|| mean-field ||

- Calogero 2-particle scattering shift



$$\phi_{ca}(w) = -\log\left(1 + \frac{1}{w^2}\right)$$

$w = \lambda_1 - \lambda_2$

|| quasi-particle ||

6. free energy functional (1 - particle)

$$\rho \geq 0 \ , \quad \int_{\mathbb{R}} dw\, \rho(w) = \frac{1}{\nu}$$

$$\mathcal{F}(\rho) = \int_{\mathbb{R}} dw\, \rho(w) \Big(V(w) - 1 + \log \rho(w) - \log \big(1 + \int_{\mathbb{R}} dw'\, \rho(w')\, \phi_{ca}(w - w') \big) \Big)$$

minimizer ρ^* (unique)

$$\mathcal{F}(\rho^*) = F(\nu, V) = \lim_{N\to\infty} -\frac{1}{N} \log Z_N(\nu, V) \qquad \Big\| \quad \nu = \frac{\ell}{N}$$

Lax DOS $\rho_Q = \nu\, \rho^*$

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hydrodynamic equation:

GHD

$$\partial_t \rho(x,t;w) + \partial_x \big(v^{\text{eff}}(x,t;w)\, \rho(x,t;w) \big) = 0$$

$$v^{\text{eff}}(w) = w + \int dw'\, \rho(w')\, \phi_{ca}(w' - w) \big(v^{\text{eff}}(w') - v^{\text{eff}}(w) \big)$$

7. Molecular dynamics (Toda approximation)

linearized GHD $\Leftrightarrow$ GGE spacetime correlator

- Toda, thermal equilibrium, $V(x) = \beta x^2$, $\beta > 0$

Grava, Kriecherbauer, McLaughlin, Mazzuca, Mendl (2023)

correlator

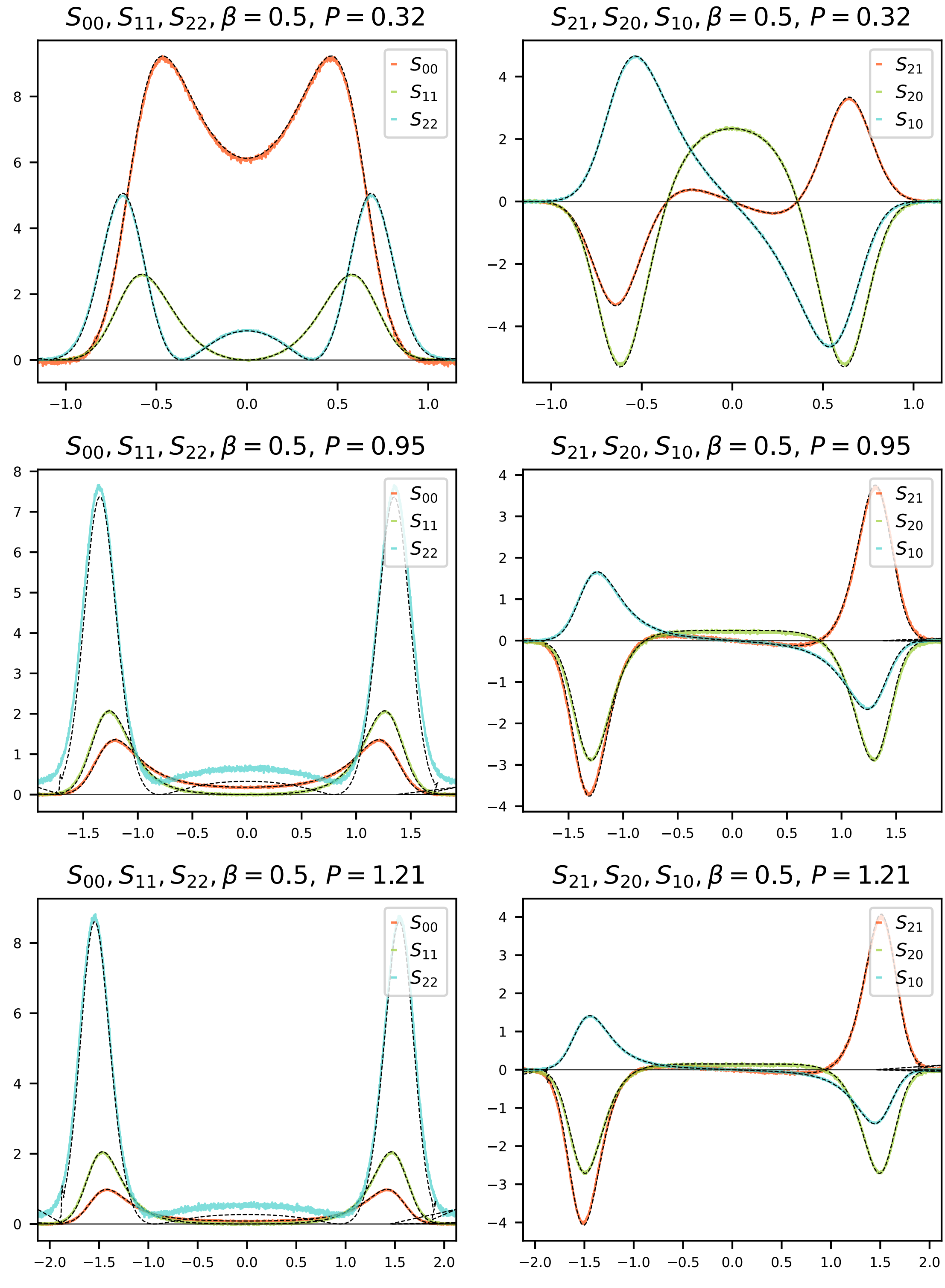
$$\langle Q^{[m]}(x,t)\, Q^{[n]}(0,0) \rangle^{c}_{\underset{\text{pressure}}{P},\beta} \cong \frac{1}{t} \underbrace{g_{m,n}}_{\text{computed from linearized GHD}}\left(\frac{1}{t}x\right), \quad m,n = 0,1,2$$

$N = 4000$, $t = 600$, $\beta = 0.5$,

$P =$	0.32	0.95	1.21
$\nu =$	2.58	-0.03	-0.42

$\underbrace{\nu = 2.58}_{\cong \text{ Calogero}}$

NO adjustable parameter !!



$S_{00}, S_{11}, S_{22}, \beta = 0.5,\ P = 0.32$
$S_{21}, S_{20}, S_{10}, \beta = 0.5,\ P = 0.32$
$S_{00}, S_{11}, S_{22}, \beta = 0.5,\ P = 0.95$
$S_{21}, S_{20}, S_{10}, \beta = 0.5,\ P = 0.95$
$S_{00}, S_{11}, S_{22}, \beta = 0.5,\ P = 1.21$
$S_{21}, S_{20}, S_{10}, \beta = 0.5,\ P = 1.21$
S_{00}
S_{11}
S_{22}
S_{21}
S_{20}
S_{10}

Outlook

- GGE spatial mixing   for Toda   Mazzuca, Memin (2023)
  exponential
- MD   density of states ✓
- rational $\frac{1}{x^2}$ scattering shift $= 0$

  $\rightsquigarrow$ $v^{\text{eff}}(w) = w$ $\partial_t \rho + \partial_x v \rho = 0$