The Cost of Stochastic Resetting

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Plan: The Cost of Stochastic Resetting

Plan

1. General idea of resetting
II. Renewal approach
III. Stochastic versus deterministic resetting
IV. Cost of resetting

References:

M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. **106**, 160601 (2011)
M. R. Evans, S. N. Majumdar, G. Schehr, J. Phys. A: Math. Theor. **53**, 193001 (2020)
J. C. Sunil, R. A. Blythe, M. R. Evans, S. N. Majumdar J. Phys. A: Math. Theor. **56**, 395001 (2023)

I Introduction: Search Problems

How does one search for lost keys?

after a while go back to where they should be and start looking again i.e. reset the search

Search Problems are ubiquitous in nature and occur in a variety of contexts

- from foraging of animals to target location on DNA
- from internet searches to the mundane task of finding one's misplaced possessions

Other contexts include: Complex chemical reactions; RESTART algorithm; Catastrophes in population dynamics

I Resetting: General Idea

Very General Idea

Resetting cuts off errant trajectories and long time tails

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Archetypal Example: Diffusion with Poissonian resetting rate

$$\frac{\partial p(x,t|x_0)}{\partial t} = D\frac{\partial^2 p(x,t|x_0)}{\partial x^2} - rp(x,t|x_0) + r\delta(x-x_0)$$

i.e. resetting rate r from all $x \neq x_0$ provides source at x_0

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i.e. resetting rate r from all $x \neq x_0$ provides source at x_0

Without resetting $r = 0$ mean time to find a target is infinite

With resetting $r > 0$ mean time to find a target is finite

II Survival probability: Renewal approach

Define $q_r(x_0, t)$ as the survival probability of the diffusive process with resetting rate r to x_0, the initial position.
and an absorbing target at the origin.

The boundary/initial conditions are $q_r(0, t) = 0$ and $q_r(x_0, 0) = 1$.

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The last renewal equation for the survival probability reads

$$(1) \quad q_r(x_0, t) = \mathrm{e}^{-rt} q_0(x_0, t) + \int_0^t \mathrm{d}t' \, r \mathrm{e}^{-rt'} q_r(x_0, t - t') q_0(x_0, t')$$

1st term is prob. of no reset and survival;
2nd term is prob. of last reset at $t - t'$ and survival until t

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This is easily solved by Laplace transform and convolution theorem

$$(2) \quad \widetilde{q}_r(x_0, s) = \frac{\widetilde{q}_0(x_0, r + s)}{1 - r\widetilde{q}_0(x_0, r + s)}$$

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Just requires Laplace transform of survival prob without reset.

Survival probability with resetting

For diffusion

$$\widetilde{q}_0(x_0, s) = \frac{1 - \mathrm{e}^{(s/D)^{1/2} x_0}}{s}$$

and we obtain

$$\widetilde{q}_r(x_0, s) = \frac{1 - \mathrm{e}^{-\alpha x_0}}{s + r\mathrm{e}^{-\alpha x_0}}$$

where

$$\alpha = \left(\frac{r + s}{D}\right)^{1/2}$$

Generally the MFPT $T = -\int_0^\infty \mathrm{d}t\, t \frac{\partial q(x_0, t)}{\partial t} = \widetilde{q}(x_0, s = 0)$ and we obtain for diffusion

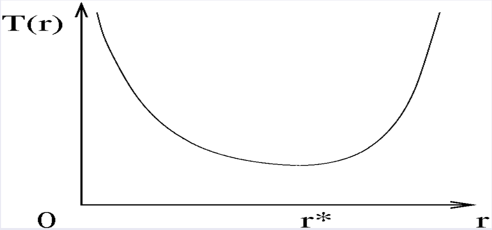
$$\boxed{T_r = \widetilde{q}_r(x_0, s = 0) = \frac{\mathrm{e}^{x_0 (r/D)^{1/2}} - 1}{r}}$$

Mean First Passage Time (MFPT)

For diffusive process with resetting

$$T_r = \widetilde{q}_r(x_0, s=0) = \frac{e^{x_0 (r/D)^{1/2}} - 1}{r}$$

$T(r)$ has a minimum at r^*



$$\frac{dT}{dr} = 0$$

$$\Rightarrow \frac{\gamma}{2} = 1 - e^{-\gamma}$$

where $\gamma = x_0 (r/D)^{1/2}$

γ= distance from target : typical distance diffused between resets

Optimal $\gamma^* = 1.5936...$

Universality at optimal resetting rate

At r^* $\boxed{\frac{\sigma(T_{r*})}{\langle T_{r^*}\rangle} = 1}$ (Reuveni PRL 2016)

This is a universal property common to all first passage time processes subject to stochastic resetting.

Moreover this provides criterion for resetting to expedite a first-passage process i.e.

$$\frac{\sigma(T)}{\langle T\rangle} > 1$$

in absence of resetting (Pal, Kostinski, Reuveni, JPA 2022)

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Roughly speaking, if this condition holds there are long tails which resetting can cut off.

III Stochastic versus Deterministic resetting

Generalise resetting rate to a waiting time distribution $\psi(t)$ between resets e.g.

$\psi(t) = r\mathrm{e}^{-rt}$ Poissonian resetting with rate r

$\psi(t) = \delta(t - t_r)$ Deterministic reset period t_r

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What is optimal choice of $\psi(t)$ to minimise time to find target?

Answer: depends on target distribution

If distance from reset position to target is known, then deterministic reset with best t_r is optimal (Pal, Kundu Evans JPA 2016, Bhat, De Bacco, Redner J Stat Mech 2016, Pal and Reuveni PRL 2017, Chechkin and Sokolov PRL 2018)

If target position is drawn from distribution exponential distribution centred at reset position then Poissonian reset is optimal (Evans and Ray 2023)

IV The Cost of Resetting

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Examples of 'Costs'

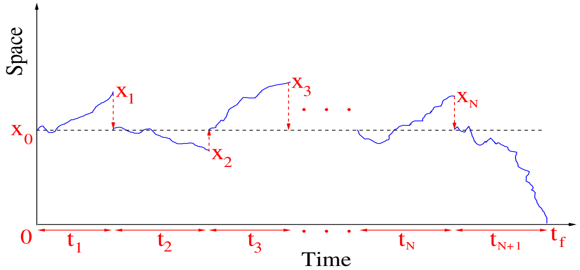
- Refractory period or fixed time penalty
  Rotbart, Reuveni, Urbakh PRE 2015, Evans and Majumdar JPA 2019
- Return phase.
  Pal, Kusmierz, Reuveni PRR 2020, Bodrova and Sokolov PRE 2020
- Energetic cost.
  Tal-Friedman *et al* JPCL 2020, Besga *et al* PRR 2020
- Thermodynamic Cost
  Fuchs, Gold, Seifert EPL 2016, Mori, Olsen, Krishnamurthy PRR 2023

IV The Cost of Resetting

Resetting shouldn't be instantaneous or free!

We impose a total cost C of resetting

$$C = \sum_{i=1}^{N} c_i \qquad\qquad c_i = c(|x_i - x_0|)$$



Natural to think of cost c_i of reset i as a time penalty.

e.g. $c_i =$ constant is fixed time penalty for reset

$c_i = \frac{1}{v}|x_i - x_0|$ is time for reset at constant speed v

Joint cost and first passage time distribution

First renewal equation for joint distribution of number of resets N, first passage time t_f and total cost $C = \sum_i c_i$

$$F(N, t_f, C|x_0) = \int_0^{t_f} \mathrm{d}t_1\, r\mathrm{e}^{-rt_1} \int_0^{\infty} \mathrm{d}x_1\, q_0(x_1, t_1|x_0) \times F(N-1, t_f - t_1, C - c_1|x_0)\Theta(C - c_1)\,,$$

t_1 is time of first reset; x_1 is position at first reset;
$c_1 = c(|x_1 - x_0|)$ is cost of first reset

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and

$$F(0, t_f, C|x_0) = \mathrm{e}^{-rt_f} F_0(t_f|x_0)\delta(C)$$

One can solve for the double Laplace transform

$$\widetilde{F}(N, s, p|x_0) = \int_0^{\infty} \mathrm{d}t_f\, \mathrm{e}^{-st_f} \int_0^{\infty} \mathrm{d}C\, \mathrm{e}^{-pC} F(N, t_f, C|x_0)$$

of this joint distribution using convolution theorem.
(Sunil, Blythe, Evans, Majumdar JPA 2023)

Joint cost and first passage time distribution: Laplace Transform Solution

$$\widetilde{F}(N,s,p) = \left[r\widetilde{W}(r+s,p|x_0)\right]^N \widetilde{F}_0(r+s|x_0)$$

where

$$\widetilde{W}(s,p|x_0) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-pc(|x-x_0|)}\, \widetilde{G}_0(x,s|x_0)$$

and $\widetilde{G}_0(x,s|x_0)$ is Laplace transform of the propagator without resetting.

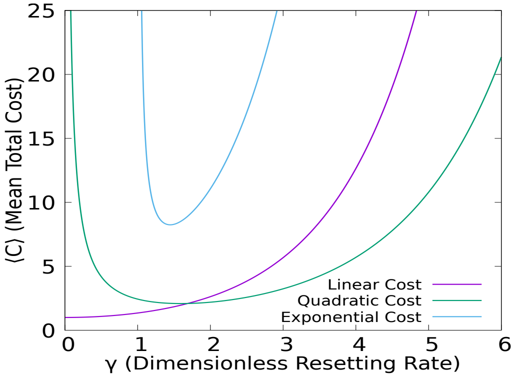
From this we obtain: distribution of N, number of resets ($p=s=0$)

$$P(N|x_0) = [r\widetilde{Q}_0(r|x_0)]^N \widetilde{F}_0(r|x_0)$$

Generating function of C, total cost ($s=0$, sum over N)

$$\langle \mathrm{e}^{-pC} \rangle = \frac{\widetilde{F}_0(r|x_0)}{1-\widetilde{W}(r,p|x_0)}$$

Mean cost



If c_i = constant, $\langle C \rangle \to 0$ as $r \to 0$

If $c_i \propto |x_i - x_0|$, $\langle C \rangle$ is *finite* as $r \to 0$

If $c_i \propto |x_i - x_0|^\beta$ with $\beta > 1$, $\langle C \rangle$ *diverges* as $r \to 0$

If $c_i \propto \exp(|x_i - x_0|/\kappa)$, $\langle C \rangle$ *diverges* at $r > 0$

Explanation of small r limit

As $r \to 0$

$$P(N|x_0) \to (1-\gamma)\gamma^N \quad \gamma = \alpha_0 x_0 \quad \text{where} \quad \alpha_0 = \left(\frac{r}{D}\right)^{1/2}$$

so $P(1|x_0) \simeq \gamma$ for small r.

Approximate position distribution at a reset

$$\begin{aligned} P(x_i) &\simeq r\int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-rt}}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x_i-x_0)^2}{4Dt}\right) \\ &= \frac{\alpha_0}{2}\mathrm{e}^{-\alpha_0|x-x_0|} \end{aligned}$$

Average cost of a reset for $c_i \propto |x_i - x_0|^\beta$

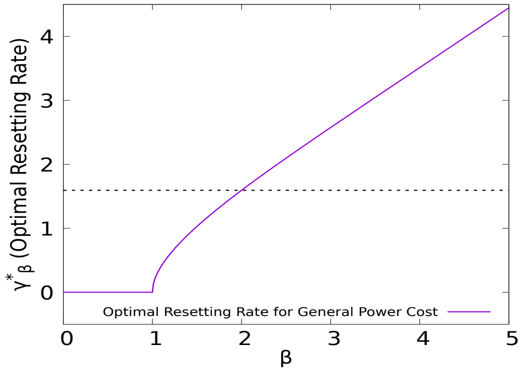
$$\langle c\rangle \sim \int_0^\infty \mathrm{d}x |x-x_0|^\beta \mathrm{e}^{-\alpha_0|x-x_0|} \sim \gamma^{-\beta}$$

So

$$\langle C\rangle \sim \gamma^{1-\beta}$$

Mean Completion Time

Now optimise the Mean Completion Time $\langle \mathcal{T} \rangle = \langle C \rangle + \langle t_f \rangle$



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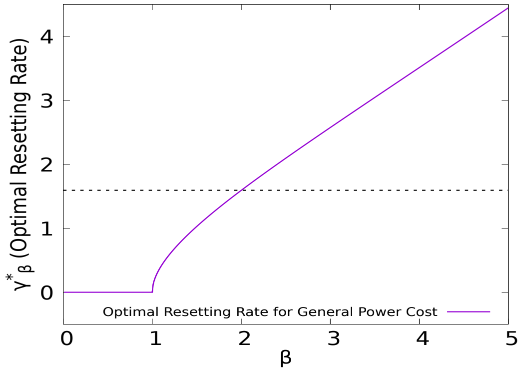

If c_i = constant or $\propto |x_i - x_0|$ optimal r is reduced from no cost case

If $c_i = \propto |x_i - x_0|^2$ optimal r is unchanged

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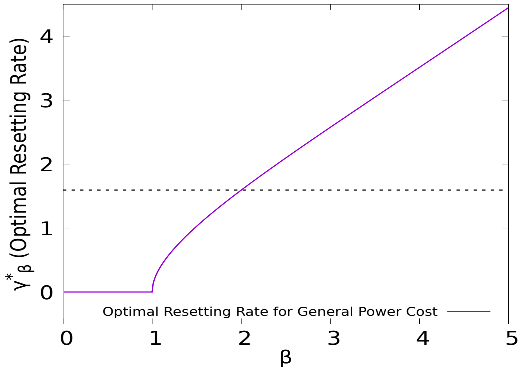

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If $c_i = \propto |x_i - x_0|^2$ optimal r is unchanged

If $c_i = \propto |x_i - x_0|^\beta$ with $\beta > 2$ optimal r is *increased*

Exponential cost distribution: condensation

If $c_i \propto \exp(|x_i - x_0|/\kappa)$ then for large C exact asymptotic behaviour is

$$P(C) \sim C^{-\kappa\alpha_0 - 1}$$

Where does this power law come from?

Exponential cost distribution: condensation

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Where does this power law come from?

Approximate position distribution at a reset

$$P(x_i) \simeq \frac{\alpha_0}{2} \mathrm{e}^{-\alpha_0|x-x_0|} \propto c_i^{-\kappa\alpha_0}$$

Then approximate distribution of cost per reset

$$f(c_i) = P(x_i) \left| \frac{\mathrm{d}x_i}{\mathrm{d}c_i} \right| \sim c_i^{-\kappa\alpha_0 - 1}$$

Condensation: One of the resets dominates total cost for large C

Summary and Conclusions

- Additive cost of resetting: $C = \sum_{i=1}^{N} c_i \qquad c_i = c(|x_i - x_0|)$.
- We have considered:
  constant, linear, power law, exponential costs $c(|x_i - x_0|)$.
- Optimisation: minimise mean total cost $\langle C \rangle$
- Optimisation: minimise mean completion time $\langle C \rangle + \langle t_f \rangle$.

  Optimal resetting rate may be decreased (constant, linear cost), unchanged (quadratic cost) or increased (superlinear cost) over no cost case
- For exponential cost, the distribution of cost per reset is a power law (long tail)

Recent developments (just a few of them)

- Resetting experiments: optical laser traps, ESE, holographic optical tweezers (Tal-Friedman et al J. Phys. Chem. Lett. 2020, Besga et al Phys. Rev. Res. 2020, JSTAT 2021, PRE 2021)
- Thermodynamics of resetting (Fuchs, Goldt, Seifert 2016)
- Fluctuation theorems for resetting (Pal & Rahav 2017, Gupta, Plata, Pal 2020)
- Quantum dynamics with stochastic reset (Rose, Touchette, Lesanovsky, Garrahan 2018, Mukherjee, Sengupta, Majumdar 2018, Das, Dattagupta, Gupta 2022, Yin & Barkai 2023)
- Run and tumble particles under resetting (Evans Majumdar 2018)
- Resetting photons (Oliveira, Maes, Meerts 2022)
- Many diffusive particles under resetting (Evans and Majumdar 2011, Vilk, Assaf, Meerson 2022, Krapivsky, Vilk, Meerson 2022)