Conical arrangement of spins in random field XY models

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Overview

- Infinite range XY Model with random conjugate fields
  1. The model
  2. 0-temperature spin distribution (perturbative regime)
  3. 0-temperature spin distribution (non-perturbative regime)
  4. Metastable states
  5. Timescales associated to the formation and orientation of cone
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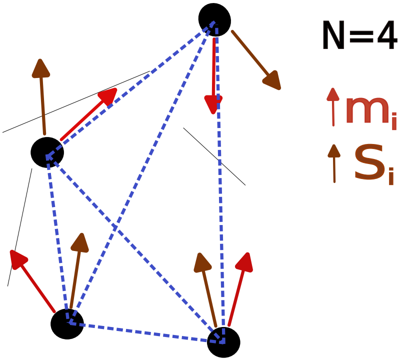
Infinite range XY model with random fields

The model



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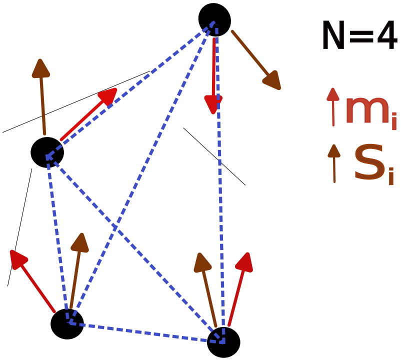

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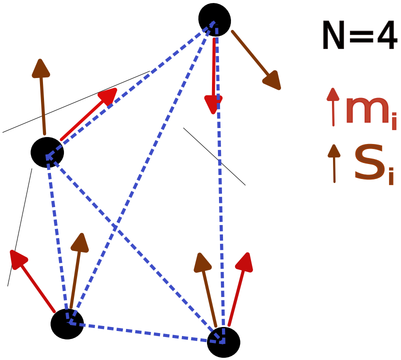

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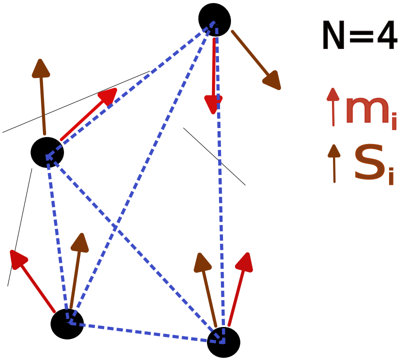

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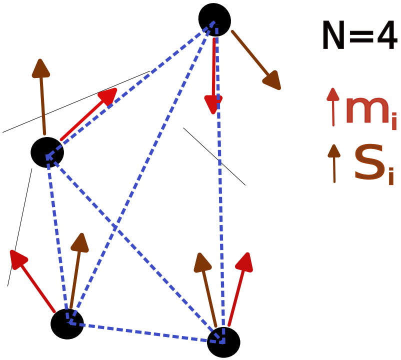

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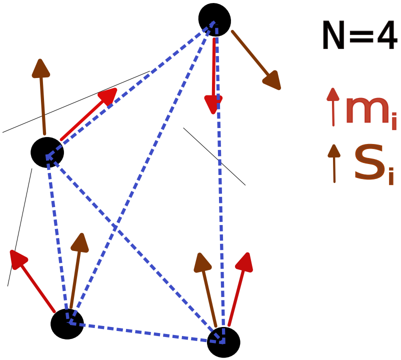

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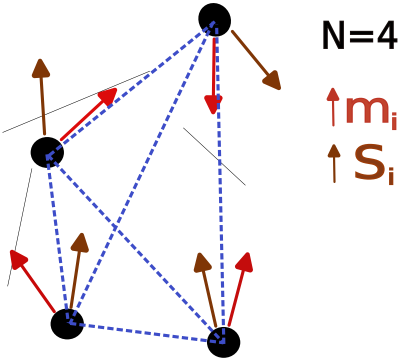

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$$H = -\frac{J}{2N}\left(\sum_{i=1}^{N}\boldsymbol{S}_i\right)^2 - h\sum_{i=1}^{N}\boldsymbol{m}_i\cdot\boldsymbol{S}_i. \tag{1}$$

- Spin models with random fields have been studied in a variety of contexts [1]
- Our focus: XY spin model with infinite range interaction in the presence of random fields. (Applications can be found in the context of nueral networks [2] etc.)
- Phase diagram in general depends on the distribution of fields. For uniform distribution see Fig. 1. Figure taken from [3]

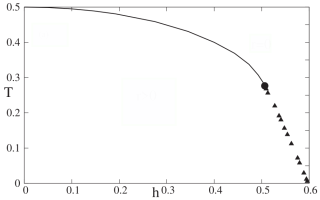


Figure: Solid line (triangled line) show locus of continuous (first order) phase transition.

We focus on 0-temperature distribution of spins

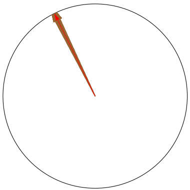
[1] **A. I. Larkin (1970).** *Soviet Journal of Experimental and Theoretical Physics.* **31**, 784; **S. Fishman and A. Aharony (1979).** *Journal of Physics C: Solid State Physics.* **12**, L729

[2] **N. Stroev and N. G. Berloff (2021).** *arXiv preprint arXiv:2103.17244*

[3] **Sumedha and M. Barma (2022).** *Journal of Physics A: Mathematical and Theoretical* **55.9**,095001

The arrangement

- In the limit $h/J \to 0$, all the spins $\boldsymbol{S}_i$ point along the direction of vector sum of fields (DVSF).



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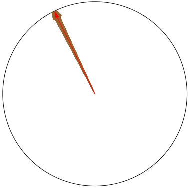
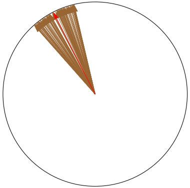


- For $h/J << 1$, all the spins fall within a cone, such that the DVSF bisect the cone.



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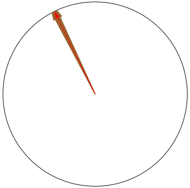

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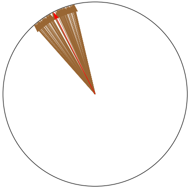
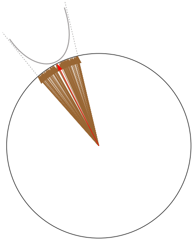


- Further, the distribution is maximum towards the edges of the cone and minimum at the centre of the cone (i.e. along the DVSF), despite the fact that all spins point along the field-axis at $h/J = 0$



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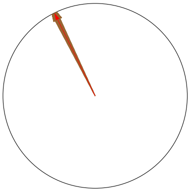

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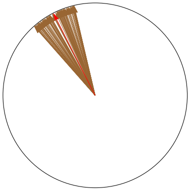

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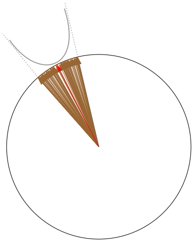
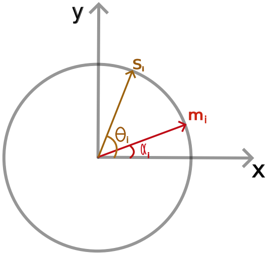


- The cone widens with h/J, and the phase transition occurs when $h/J \simeq 0.64$ and the cone angle $\phi_c \sim 90^o$.

0-temperature distribution of spins (perturbative regime)

$$H = -\frac{J}{2N}\left(\sum_{i=1}^{N} \boldsymbol{S}_i\right)^2 - h\sum_{i=1}^{N} \boldsymbol{m}_i \cdot \boldsymbol{S}_i.$$



$$H = -\frac{J}{2N}\left[\sum_{i=1}^{N} (\cos(\theta_i)\,\hat{\boldsymbol{x}} + \sin(\theta_i)\,\hat{\boldsymbol{y}})\right]^2 - h\sum_{i=1}^{N} \cos(\theta_i - \alpha_i).$$

- At $h/J = 0$, $\theta_i = \theta^{(0)}$ minimizes the energy.
- Setting $\partial H/\partial \theta_i = 0$, yields the following equations for extrema

$$\frac{J}{N}\sum_{j=1}^{N} \sin(\theta_i - \theta_j) + h\sin(\theta_i - \alpha_i) = 0. \tag{2}$$

for $i = 1, 2, ..., N$

The equation for extrema is perturbatively solved by expanding θ_i as:

$$\theta_i = \theta^{(0)} + \frac{h}{J}\,\theta_i^{(1)} + \frac{h^2}{J^2}\,\theta_i^{(2)} + \dots$$

To first order in h/J we have

$$\sum_{j=1}^{N} \left(\theta_i^{(1)} - \theta_j^{(1)}\right) + N\frac{h}{J}\sin(\theta^{(0)} - \alpha_i) = 0. \tag{3}$$

Summing the above over i yields

$$\sum_{i=1}^{N} \sin(\theta^{(0)} - \alpha_i) = 0 \implies \tan(\theta^{(0)}) = \frac{\sum_{j=1}^{N} \sin(\alpha_i)}{\sum_{j=1}^{N} \cos(\alpha_i)} = \tan(\alpha_0). \tag{4}$$

where α_0 is equal to the angle that the sum of the random fields, $\boldsymbol{h} = h\sum_{j=1}^{N} \boldsymbol{m}_j$, makes with the x-axis.

Equation (3) is solved by

$$\theta_i^{(1)} = \sin(\alpha_i - \theta^{(0)}). \tag{5}$$

Hence, we have

$$\theta_i = \theta^{(0)} + \frac{h}{J}\sin(\alpha_i - \theta^{(0)}) \tag{6}$$

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Let $\widetilde{P}$ and P be the probability distribution functions for α_i and $\Delta\theta_i = \theta_i - \theta^{(0)}$, respectively. If

$$\widetilde{P}(\alpha_i) = \frac{1}{2\pi}, \tag{7}$$

then

$$P(\Delta\theta_i) = \frac{1}{\pi\sqrt{(h/J)^2 - \Delta\theta_i^2}} \tag{8}$$

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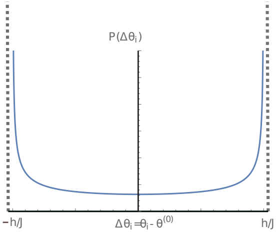

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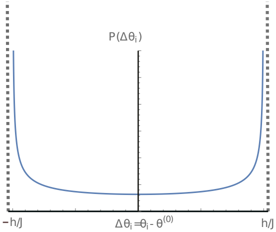

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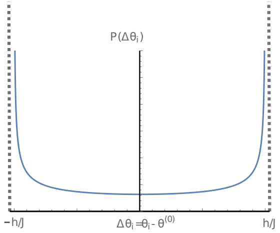

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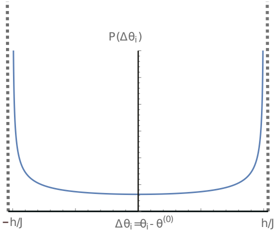

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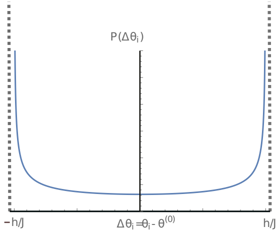

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- The cone angle ϕ_c, defined here as the angular separation between the maxima of the distribution (8), is given by

$$\phi_c = \frac{2h}{J}. \tag{9}$$

Numerical continuation to higher values of h/J

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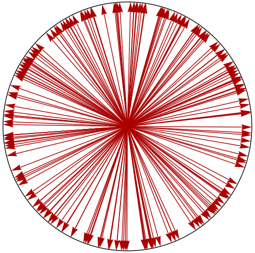
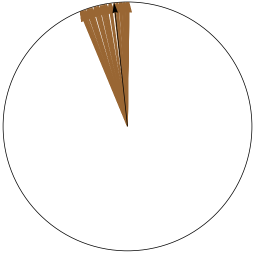
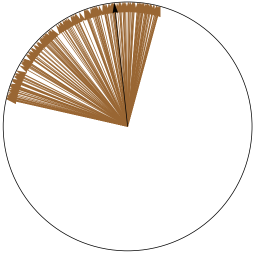


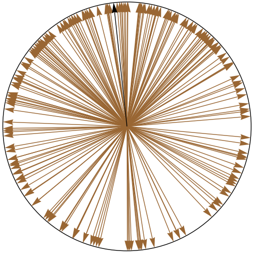
(a) Directions of the quenched random fields



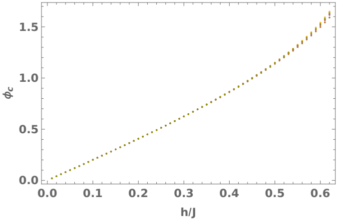
(b) Directions of spins $\boldsymbol{S}_i$ at $h/J = 0.2$



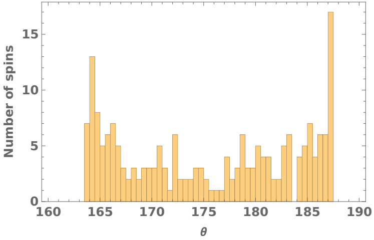
(c) Directions of spins $\boldsymbol{S}_i$ at $h/J = 0.62$



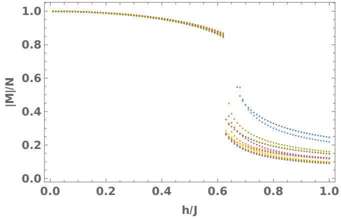
(d) Directions of spins $\boldsymbol{S}_i$ at $h/J = 0.63$



(e) Cone angle ϕ_c vs h/J for $N = 200$



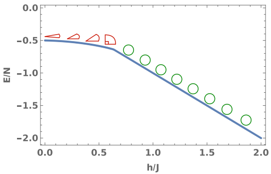
(f) Histogram of distribution of spins $\boldsymbol{S}_i$ at $h/J = 0.2$.



(g) Magnetization $|\sum_i \boldsymbol{S}_i|/N$ vs h/J.

Metastable states

- Energy of minima-states as a function of h/J is shown in the right



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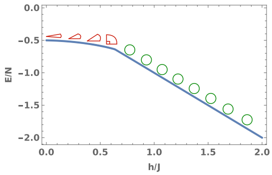
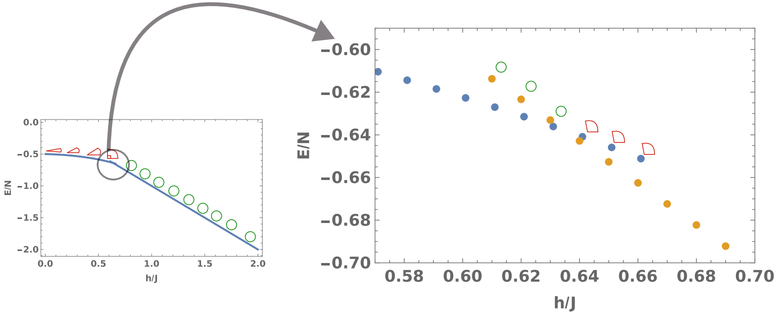
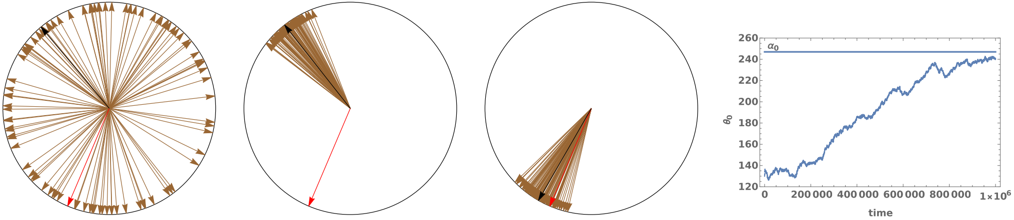


- Apart from the minima-states there are metastable states near the phase transition



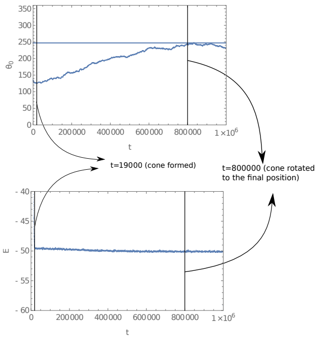
Cone formation and orientation times (Monte-Carlo simulation)



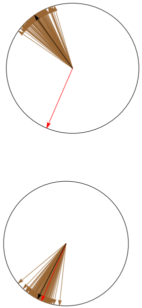
(h) Spins at $t = 0$ (i) Spins at t=19000 (j) Spins at $t = 10^6$ (k) θ_0 vs time

Figure: Results of MC simulations. Number of spins $N = 100$. The black arrow shows the direction of the sum of spins and the red arrow that of fields. $h/J = 0.1$ and temperature $T = 0.01$

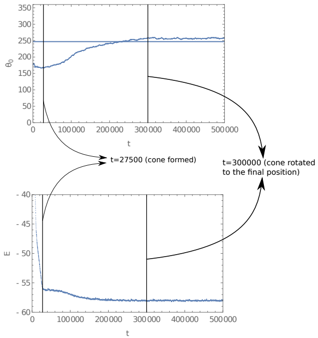
- There are two time scales in the system
- Time taken for the formation of cone $t_{cf} \sim 1.9 \times 10^4$ (increases with h/J).
- Time taken for the cone to orient in the right direction t_{co} which varies with initial condition. For the above case $\sim 10^6$ (decreases with h/J).



Top: Angle of vector sum of spins θ_0 vs time t
Bottom: Energy E vs time t



Top: Spins at t=19000
Bottom: Spins at t=800000
Red arrow shows the direction of vector sum of fields
Blackarrow shows the direction of vector sum of spins



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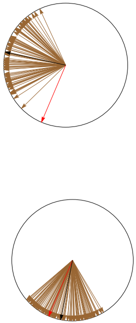


Top: Spins at t=27500
Bottom: Spins at t=300000
Red arrow shows the direction of vector sum of fields
Blackarrow shows the direction of vector sum of spins

h/J = 0.1 and temperature = 0.01

h/J = 0.5 and temperature = 0.01

Large time evolution

$$H = -\frac{J}{2N}\left(\sum_{i=1}^{N} \boldsymbol{S}_i\right)^2 - h\sum_{i=1}^{N} \boldsymbol{m}_i \cdot \boldsymbol{S}_i. \tag{10}$$

- Set $\boldsymbol{S}_i = \boldsymbol{S}_0$. Then,

$$H = constant - \boldsymbol{S}_0 \cdot \underbrace{h\sum_{i=1}^{N} \boldsymbol{m}_i}_{\boldsymbol{h}_0} = constant - \boldsymbol{S}_0 \cdot \boldsymbol{h}_0 \tag{11}$$

- Let $\boldsymbol{S}_0$ make an angle θ_0 with the x-axis, and $\boldsymbol{h}_0$ and angle α_0. We can write a dynamical equation for θ_0:

$$\frac{d\theta_0}{dt} = -\gamma\frac{\partial H}{\partial \theta_0} \qquad \Longrightarrow \quad \frac{d\theta_0}{dt} = -\gamma|\boldsymbol{h}_0|\sin(\theta_0 - \alpha_0) \tag{12}$$

- There is time scale $1/(\gamma|\boldsymbol{h}_0|)$ which decreases with the magnitude $|\boldsymbol{h}_0|$.

  Note that $|\boldsymbol{h}_0|$ quantifies how uneven the distribution of fields is.

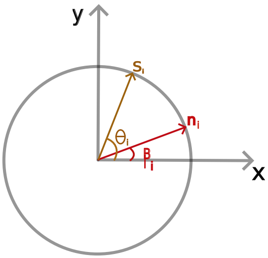
Infinite range XY model with random crystal field

$$H = -\frac{J}{2N}\left(\sum_{i=1}^{N} \boldsymbol{S}_i\right)^2 - D\sum_{i=1}^{N}(\boldsymbol{n}_i \cdot \boldsymbol{S}_i)^2$$

- Studied in the context of amorphous magnets [4].
- Phase diagram is sensitive to the distribution of crystal fields [5].
- There is no phase transition at 0-temperature for uniform distribution of fields. [6]

[4] **R. Harris, M. Plischke, and M. J. Zuckermann (1973)** *Physical Review Letters* **31,** 160.
[5] **K. H. Fischer and A. Zippelius (1985).** *Journal of Physics C: Solid State Physics.* **18**, 1139.
[6] **Sumedha and M. Barma (2022).** *Physical Review E* **105,** 024111.

0-temperature distribution of spins (perturbation theory)



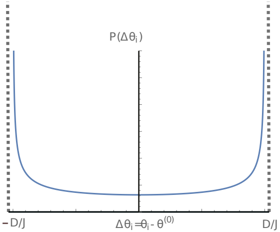
$$H = -\frac{J}{2N}\left[\sum_{i=1}^{N}\left(\cos(\theta_i)\,\hat{\mathbf{x}} + \sin(\theta_i)\,\hat{\mathbf{y}}\right)\right]^2 - D\sum_{i=1}^{N}\cos^2(\theta_i - \beta_i).$$

$$\frac{\partial H}{\partial \theta_i} = 0 \implies \frac{J}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j) + D\sin(\theta_i - \beta_i) = 0.$$

- For $D/J << 1$

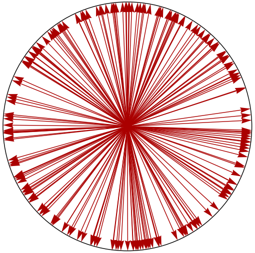
$$\theta_i = \theta^{(0)} + \frac{D}{J}\sin\left[2(\beta_i - \theta^{(0)})\right],$$
$$\tan(2\theta^{(0)}) = \frac{\sum_{i=1}^{N}\sin(2\beta_i)}{\sum_{i=1}^{N}\cos(2\beta_i)}. \quad (13)$$

$$P(\Delta\theta_i) = \frac{1}{\pi\sqrt{(D/J)^2 - \Delta\theta_i^2}}$$

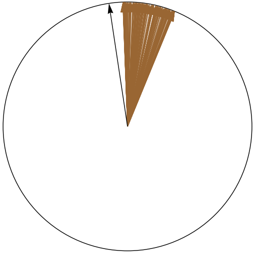


- For small values of D/J, the spin angles θ_i are distributed within a two dimensional cone.
- The distribution is maximum towards the edges of the cone marked by the spins for which $\Delta\theta_i = \pm D/J$.
- Within the cone, the distribution is minimum at the center, when $\Delta\theta_i = 0$.
- The cone angle ϕ_c, defined here as the angular separation between the maxima of the distribution (8), is given by

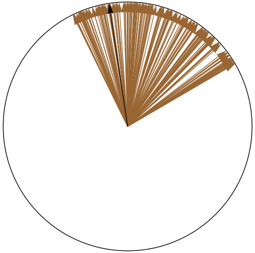
$$\phi_c = \frac{2D}{J}. \tag{14}$$



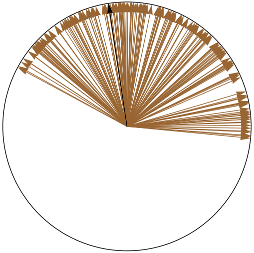
(a) Random fields $\boldsymbol{m}_i$



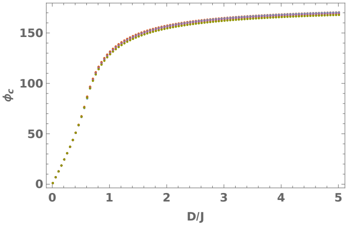
(b) Spins $\boldsymbol{S}_i$ at $D/J = 0.2$



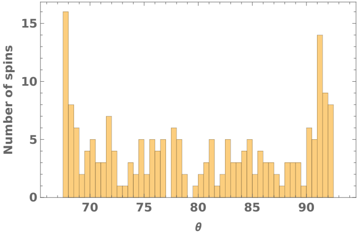
(c) Spins $\boldsymbol{S}_i$ at $D/J = 0.61$



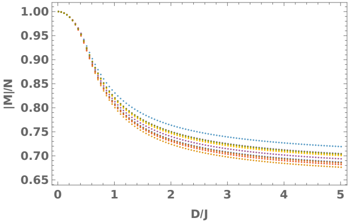
(d) Spins $\boldsymbol{S}_i$ at $D/J = 2.21$



(e) Cone angle ϕ_c (in degrees) vs D/J.



(f) Histogram of distribution of spins $\boldsymbol{S}_i$ at $D/J = 0.21$.



(g) The order parameter $|\boldsymbol{M}|/N$ vs D/J.

Summary

Infinite range XY model with random field

- Arrangement of spins in the ordered phase (within a cone).
- Phase transition occurs when $\phi_c \sim 90^o$
- Existence of metastable states
- Cone formation and cone orientation times.

Summary

Infinite range XY model with random field

- Arrangement of spins in the ordered phase (within a cone).
- Phase transition occurs when $\phi_c \sim 90^o$
- Existence of metastable states
- Cone formation and cone orientation times.

Infinite range XY model with random crystal field

- 0-temperature arrangement of spins (within a cone)
- There is no phase transition
- Spins span a hemisphere when $D \to \infty$

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