Inclusions in momentum conserving active fluids

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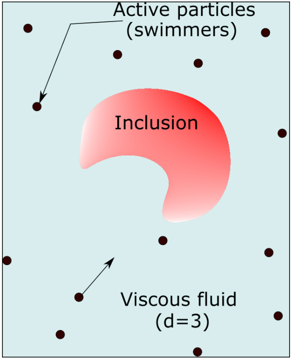
Joint work with: Yariv Kafri & Sriram Ramaswamy



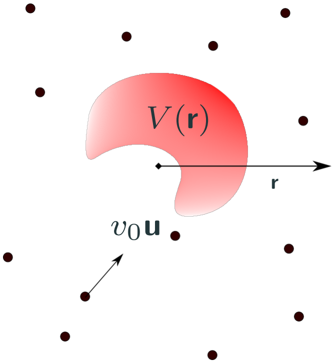
Dry active systems

Inclusion in a dry active system

Equations of motion

$$\dot{\mathbf{x}}_i = v_0 \mathbf{u}_i - \mu \boldsymbol{\nabla} V(\mathbf{x}_i)$$

- $v_0 \mathbf{u}$ self-propulsion force, correlation time τ
- $V(\mathbf{x})$ localized external potential
- no hydrodynamic interactions



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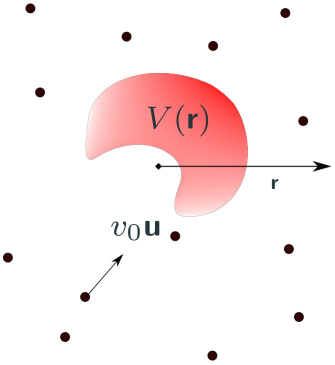

In equilibrium, Boltzmann distribution

$$\rho(\mathbf{x}) = \rho_0 \exp(-\beta V(\mathbf{x}))$$

$\Rightarrow$ local effect of the potential

Non-equilibrium dynamics: **universal** long-range influence of a localized object Y. Baek & al. PRL (2018)

Far-field density decay

$$D\Delta\rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{\nabla} V) = -\boldsymbol{\nabla} \cdot \mathbf{j}$$

- diffusion on large scales
- interaction with the potential
- $\mathbf{j}$ accounts for all complicated near-field effects (unknown)

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Far-field solution via a multipole expansion

$$\rho(\mathbf{x}) \simeq -\frac{1}{4\pi D}\frac{x^\alpha}{x^3}\int \mathrm{d}\mathbf{x}' \, \left(j^\alpha(\mathbf{x}') + \rho(\mathbf{x}')\partial^\alpha V(\mathbf{x}')\right)$$
$$= O\left(x^{-2}\right)\,, \quad \text{universal decay}$$

Far-field density decay

Consequences

- effective long-range interactions between passive bodies embedded in active systems
- strong influence of bulk and boundary disorder on phase behavior

Open question

- what happens when self-propelled particles move in a fluid and momentum is locally conserved?

Goal for today

- find the far-field density profile for swimmers in a viscous fluid with a localized obstacle

Challenge

- long-range interactions mediated by the fluid

Viscous flows

Viscous flows

For microswimmers $\Rightarrow$ viscous flow

$$\text{Reynolds} = \frac{\text{inertia}}{\text{viscosity}} \ll 1 \;\; \text{for instance} \sim 10^{-5} \;\; \text{for E. Coli}$$

Stokesian incompressible fluid

$$\eta \boldsymbol{\nabla}^2 \mathbf{v} - \boldsymbol{\nabla} P + \mathbf{f}(\mathbf{r}) = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

- $\boldsymbol{\nabla}^2 \mathbf{v}$: diffusion of momentum
- $\mathbf{f}(\mathbf{r})$: momentum sources

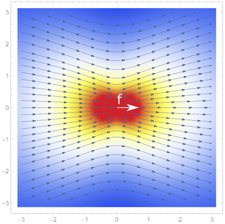
$\Rightarrow$ local perturbations induce long-range flows

Fundamental solutions

- Force monopole: $\mathbf{f}(\mathbf{r}) = \mathbf{f}\delta(\mathbf{r})$

$$\eta \boldsymbol{\nabla}^2 \mathbf{v} - \boldsymbol{\nabla} P + \mathbf{f}\delta(\mathbf{r}) = 0$$

$$v^\alpha(\mathbf{r}) = \frac{1}{8\pi\eta} J^{\alpha\beta}(\mathbf{r}) f^\beta \text{ with } J^{\alpha\beta}(\mathbf{r}) = \frac{\delta^{\alpha\beta}}{r} + \frac{r^\alpha r^\beta}{r^3} \sim \frac{1}{r}$$



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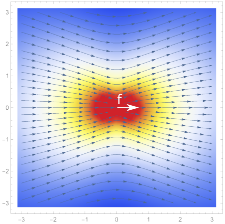
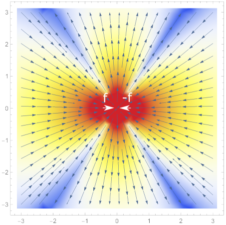


- Force dipole:

$$\mathbf{f}(\mathbf{r}) = \mathbf{f}\delta(\mathbf{r} + \mathbf{a}) - \mathbf{f}\delta(\mathbf{r}) \simeq \mathbf{f}\left(\mathbf{a} \cdot \boldsymbol{\nabla}\delta(\mathbf{r})\right)$$

$$\Rightarrow \; v^\alpha(\mathbf{r}) \simeq \frac{1}{8\pi\eta} \partial_\mu J^{\alpha\beta}(\mathbf{r}) Q^{\mu\beta} \sim \frac{1}{r^2}$$



with dipole strength $Q^{\mu\beta} = a^\mu f^\beta$.

Inclusion in a momentum conserving active fluid

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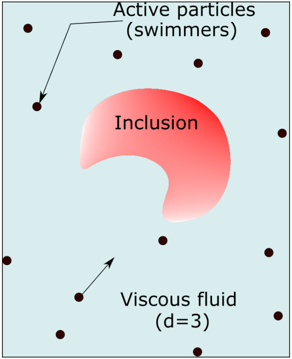
- no orentational order in the bulk $\Rightarrow$ bulk fluid flow vanishes on average
- no large scale convective transport in the bulk
- large-scale diffusion of swimmers in the bulk
- slowly-decaying average flow far-away from the obstacle



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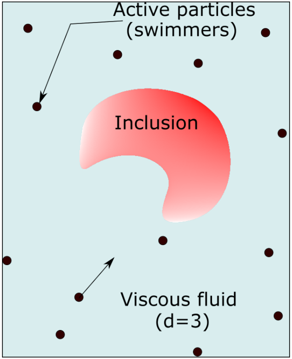

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Inclusion in a momentum conserving active fluid

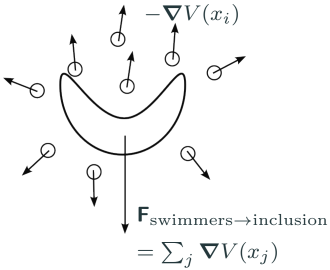
$$D\Delta\rho + \boldsymbol{\nabla}\cdot\left(\rho\left[\mathbf{v} + \boldsymbol{\nabla}V\right]\right) = -\boldsymbol{\nabla}\cdot\mathbf{j}$$

- derive it from a microscopic model of swimmers
- obtain the form of the effective velocity $\mathbf{v}(\mathbf{r})$
- solve this equation in the far-field

Microscopic model & velocity field

Many-body equations of motion

- active particles (squirmers): spheres with a given surface flow $\mathbf{v}_s(\mathbf{x})$
- inclusion exerts a short range force $-\boldsymbol{\nabla} V(\mathbf{x}_i)$ on each swimmer
- inclusion is an impenetrable object



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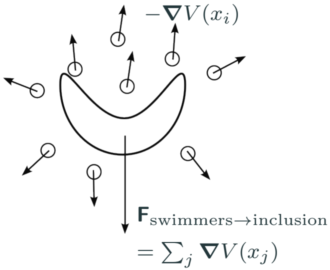

Stokes equation

$$\eta \Delta \mathbf{v} - \boldsymbol{\nabla} P = 0\,, \quad \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

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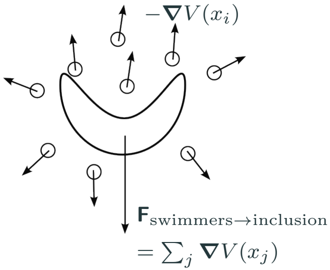

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Boundary conditions on the swimmers

$$\mathbf{v}(\mathbf{x})\big|_i = \dot{\mathbf{x}}_i + \mathbf{v}_{s,i}(\mathbf{x})\,.$$

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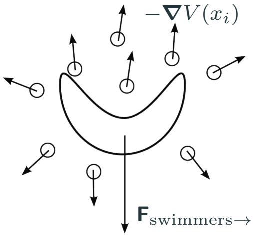

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overdamped equations of motion

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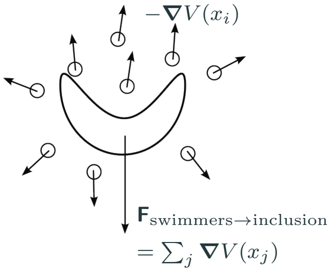

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overdamped equations of motion

No-slip boundary condition on the surface of the obstacle

Far-field fluid flow

- each swimmer close to the obstacle behaves as a force monopole $\mathbf{F}_{\text{swimmer}\to\text{fluid}} = -\boldsymbol{\nabla}_{\mathbf{x}_i} V(\mathbf{x}_i)$
- the obstacle behaves as a force monopole $\mathbf{F}_{\text{obs}\to\text{fluid}}$

Total force monopole

$$\mathbf{F}_{\text{obs}\to\text{fluid}} - \sum_i \boldsymbol{\nabla}_{\mathbf{x}_i} V(\mathbf{x}_i) = -\left(\mathbf{F}_{\text{fluid}\to\text{obs}} + \mathbf{F}_{\text{swimmers}\to\text{obs}}\right)$$

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- Freely-moving obstacle: conservation of momentum $\Rightarrow$ zero monopole

$$v^\alpha(\mathbf{r}) \simeq \frac{1}{8\pi\eta}\partial_\gamma J^{\alpha\beta}(\mathbf{r}) Q_{\mathrm{eff}}^{\gamma\beta} \sim \frac{1}{r^2}$$

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- Fixed obstacle: source of momentum $\Rightarrow$ non-zero monopole

$$v^{\alpha}(\mathbf{r}) \simeq \frac{1}{8\pi\eta} J^{\alpha\beta}(\mathbf{r}) \left\langle F_{\text{ext}}^{\beta} \right\rangle \sim \frac{1}{r}$$

where $\left\langle F_{\text{ext}}^{\beta} \right\rangle$ is the average force exerted by an external observer to keep the obstacle fixed

Far-field density

Power counting

$$D\Delta\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = -\boldsymbol{\nabla}\cdot\mathbf{j}$$

In the far-field

- localized near-field effects $\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r}) \;\to \mathbf{j}\cdot\boldsymbol{\nabla}\delta(\mathbf{r})$
- $v(\mathbf{r}) \to \mu r^{-\delta} g(\hat{\mathbf{r}})$
- $\delta\rho(\mathbf{r}) = \rho(\mathbf{r}) - \rho_0 \to 0$ at large distances

$$D\Delta\delta\rho + \mu\boldsymbol{\nabla}\cdot\left(r^{-\delta} g(\hat{\mathbf{r}})\delta\rho\right) = -\mathbf{j}\cdot\boldsymbol{\nabla}\delta(\mathbf{r})$$

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$$D\Delta'\delta\rho' + \mu b^{1-\delta}\,\boldsymbol{\nabla}\cdot\left(r'^{-\delta}g(\hat{\mathbf{r}})\delta\rho'\right) = -\mathbf{j}\cdot\boldsymbol{\nabla}'\delta(\mathbf{r}')$$

Hydrodynamic coupling

- irrelevant for a freely-moving obstacle ($\delta = 2$)
- marginal for a fixed obstacle ($\delta = 1$)

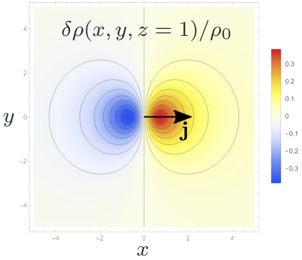
Freely-moving obstacle

Hydrodynamics irrelevant $\Rightarrow$ same behavior as in the dry case

$$\delta\rho(\mathbf{r}) \propto \frac{\hat{\mathbf{r}} \cdot \mathbf{j}}{r^2}$$

and **J** depends on near-field properties

- universal -2 decay
- universal angular dependence



Fixed obstacle

Marginal coupling to hydrodynamics ⇒ **universal** anomalous exponent and modified angular dependence

- depends on the far-field properties of the flow

$$v^{\alpha}(\mathbf{r}) \simeq \frac{1}{8\pi\eta} J^{\alpha\beta}(\mathbf{r}) \left\langle F^{\beta}_{\mathrm{ext}} \right\rangle$$

$$\Rightarrow \text{two parameters} \begin{cases} \lambda = |\langle \mathbf{F}_{\mathrm{ext}} \rangle| \,/\, (8\pi\eta D) \\ \mathbf{p} = \langle \mathbf{F}_{\mathrm{ext}} \rangle \,/\, |\langle \mathbf{F}_{\mathrm{ext}} \rangle| \end{cases}$$

- depends on the symmetry of the obstacle

Fixed obstacle: decay

For an obstacle with an axis of symmetry (with axis $\hat{\mathbf{p}}$)

$$\delta\rho(\mathbf{r}) \propto \frac{1}{r^{2+\epsilon_\parallel}} \quad \text{with} \quad \epsilon_\parallel = \frac{\lambda^2}{3} + O\left(\lambda^4\right)$$

$\Rightarrow$ **faster** decay than in the freely-moving case

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For a less symmetric obstacle

$$\delta\rho(\mathbf{r}) \propto \frac{1}{r^{2+\epsilon_{\perp}}} \quad \text{with} \quad \epsilon_{\perp} = -\frac{\lambda^2}{12} + O\left(\lambda^4\right)$$

$\Rightarrow$ **slower** decay than in the freely-moving case

Fixed obstacle: angular dependence

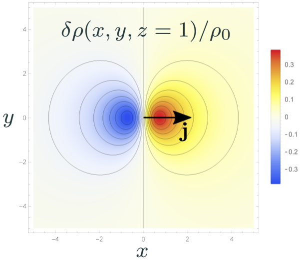
For an obstacle with an axis of symmetry (with axis $\hat{\mathbf{p}}$) $\Rightarrow$ angle θ ($\cos\theta = \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}$)

$$\delta\rho(\mathbf{r}) \propto \frac{1}{r^{2+\epsilon_{\parallel}}} g_{\parallel}(\theta)$$

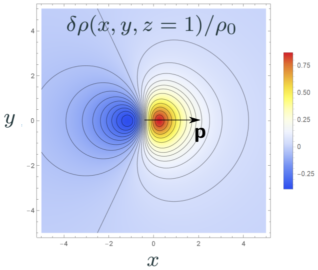
with

$$g_{\parallel}(\theta) = \cos\theta + \frac{\lambda}{4}\left(3 - 5\cos^2\theta\right) + \frac{3}{4}\lambda^2\cos^3\theta + O\left(\lambda^3\right)$$

Freely moving



Fixed, symmetric



Fixed obstacle: angular dependence

For an obstacle with no axis of symmetry $\Rightarrow$ polar angle θ ($\cos\theta = \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}$) and azimutal angle ϕ

$$\delta\rho(\mathbf{r}) \propto \frac{1}{r^{2+\epsilon_\perp}} g_\perp(\theta, \phi)$$

with

$$g_\perp(\theta, \phi) = \cos(\phi + \phi_0)\sin(\theta)\left(1 - \frac{5\lambda}{4}\cos\theta + \frac{3}{4}\lambda^2\cos^3\theta\right) + O\left(\lambda^3\right)$$

and ϕ_0 is a non-universal phase.

Conclusion

- long-range influence of an inclusion on a momentum conserving active fluid
- long-range velocity flow
- long-range density field
- decay exponent and angular dependence as a function of simple properties of the object

- interactions between passive bodies?
- influence of disorder on the bulk behavior of the system?

Thank you!