Optimizing random searches under a time constraint using Lévy flights

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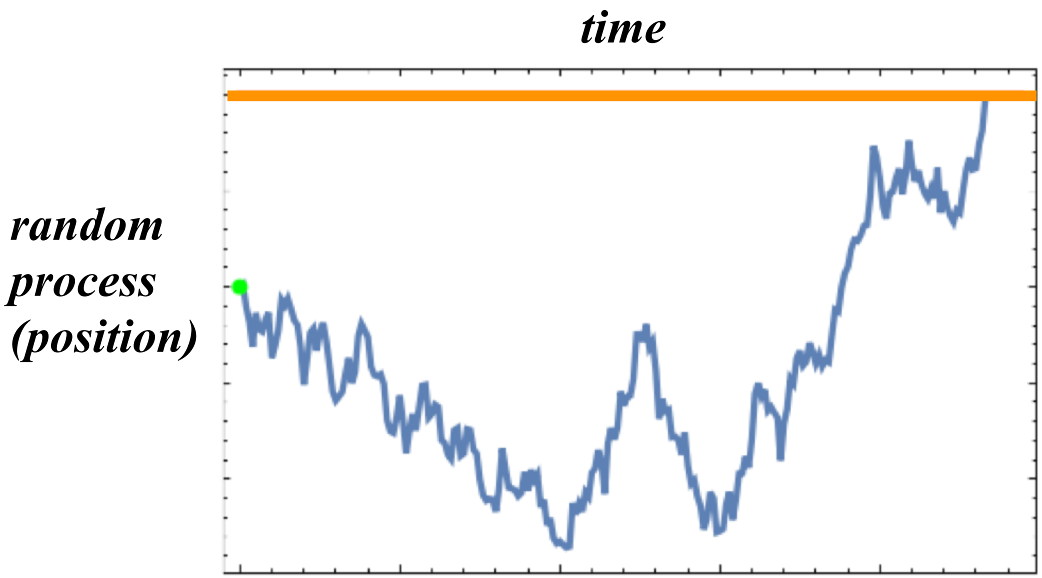
Frontiers in Statistical Physics, Raman Research Institute, Bangalore, 4-8 December 2023

First passage processes



Chemical reaction kinetics

Foraging animals

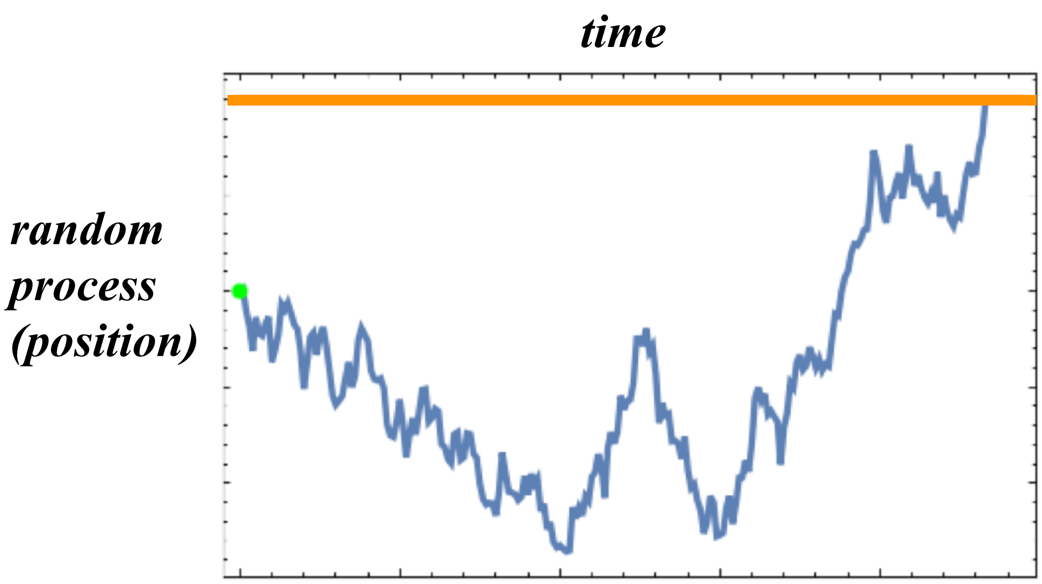
Seed dispersal

Finance

Rescue operations

….

First passage processes



Random movement of a "searcher", static target.

Chemical reaction kinetics

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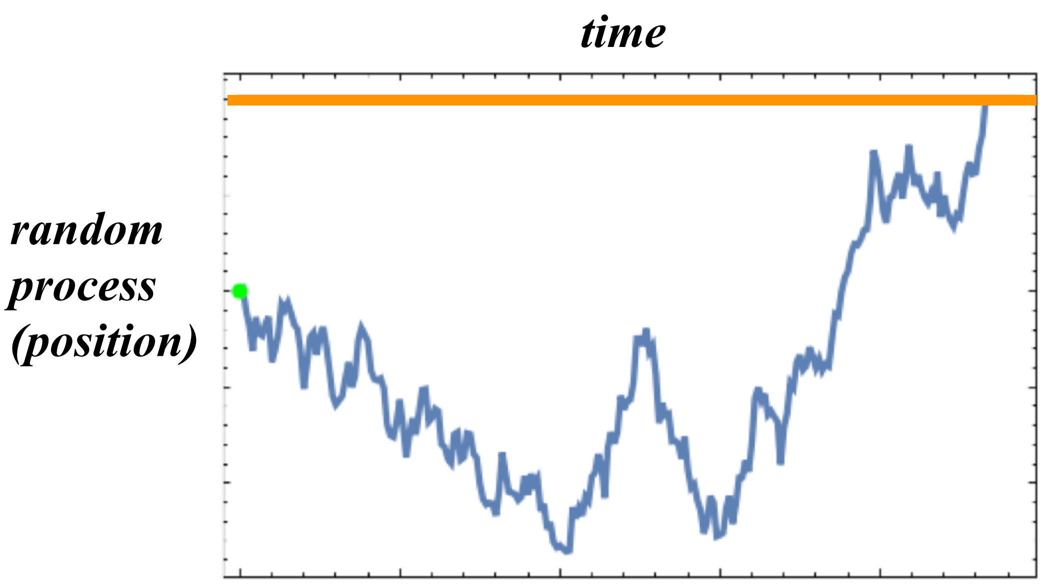
….

First passage processes



Random movement of a "searcher", static target.

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....

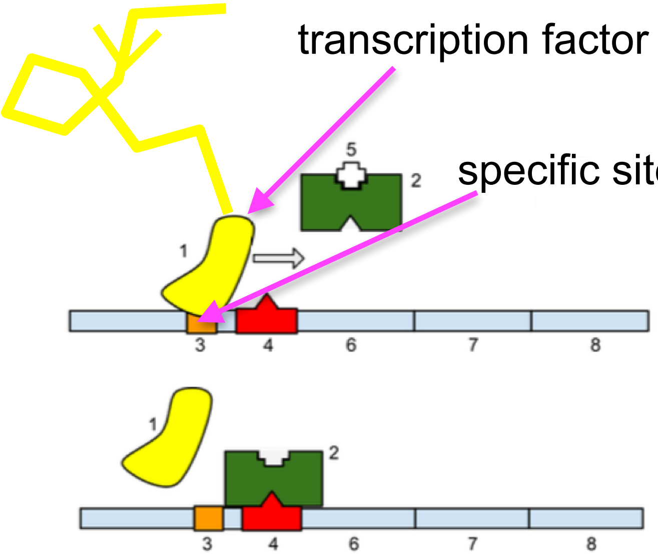
stochastic internal target dynamics

Random search processes with a time constraint

1) Gene regulation

transcription factor (searcher)

specific site on DNA (target)



(from Bénichou et al. RMP 2011)

2) Rescue

Sea operations
(Kosmas et al.,
Prod. Oper. Manag. 2022)

3) Animals with ephemeral resources

ripe fruit → rotten fruit

4) Give-up time

Search vehicle with limited autonomy
(Waharte el al. IEEE 2010)

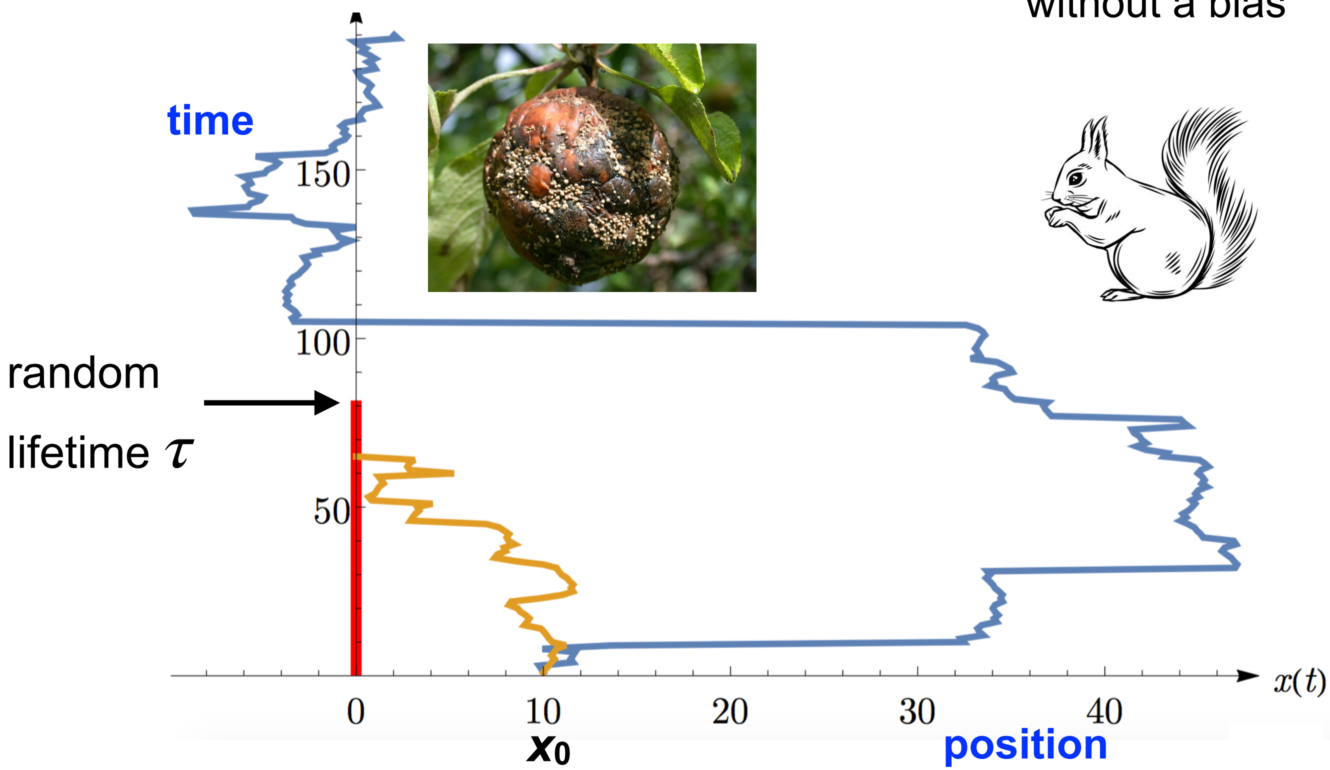
Marginal value theorem
(Charnov, Theor. Pop. Biol., 1976)

Mortal searcher (Yuste el al. PRL 2013)

Resetting processes
(Evans & Majumdar, PRL 2011)

Search of a finite-lived target

Random walk
without a bias



Search of a finite-lived target

Random walk
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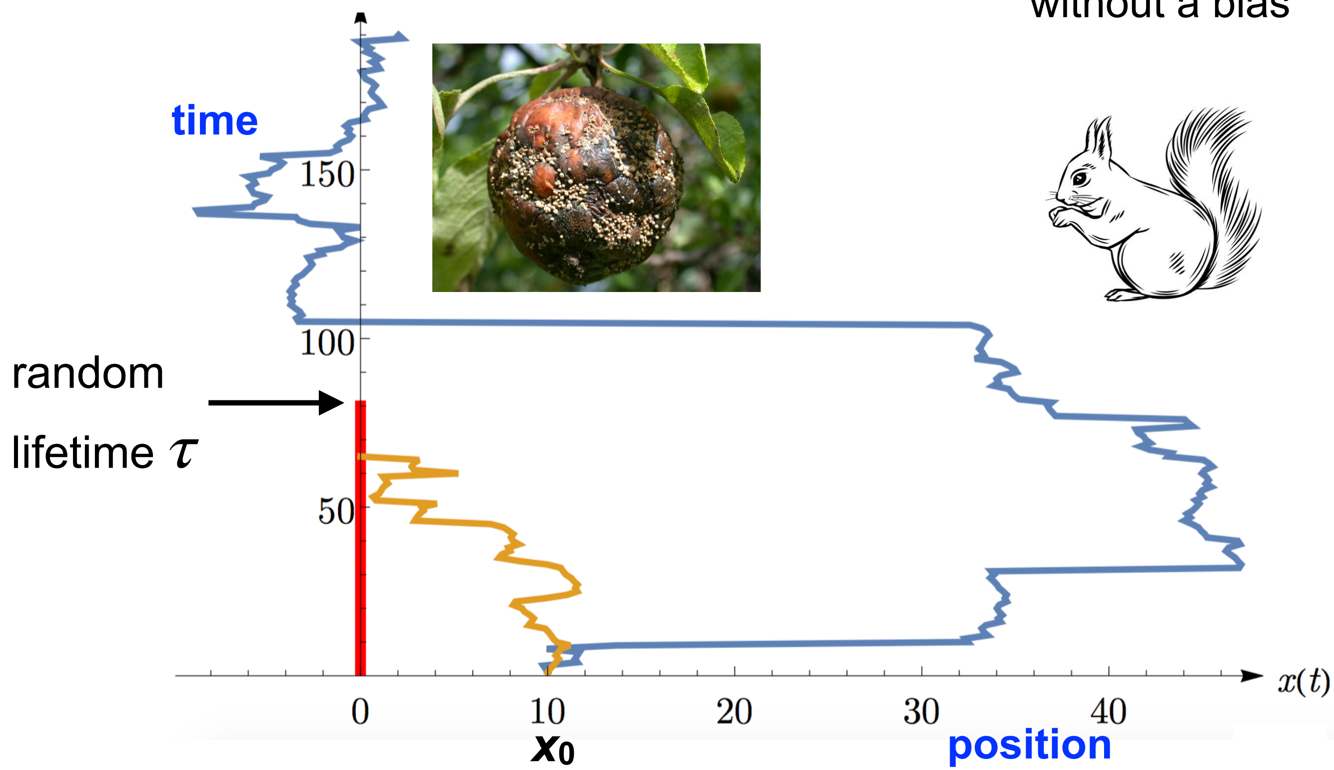


Capture probability?

Conditional mean
search time?

Optimal strategy to follow?

Dependence on the target
lifetime, initial position,…?

Brownian limit (Meerson & Redner, PRL 2015):

capture prob. $= e^{-\sqrt{\frac{\alpha}{D}}x_0}$ and CMFPT $= \dfrac{x_0}{2\sqrt{D\alpha}}$

$D \to \infty$ optimal

α : mortality rate

Outcome of a process failure/success improved by resetting (Belan, PRL 2018)

(if mortality rate is small enough)

Survival probability of a permanent target surrounded by a sea of mortal random walkers
(Yuste, Abad, Lindenberg, PRL 2013):

Mortal sub-diffusive searchers (Yuste, Abad, Yuste,Lindenberg, PRE 2012):

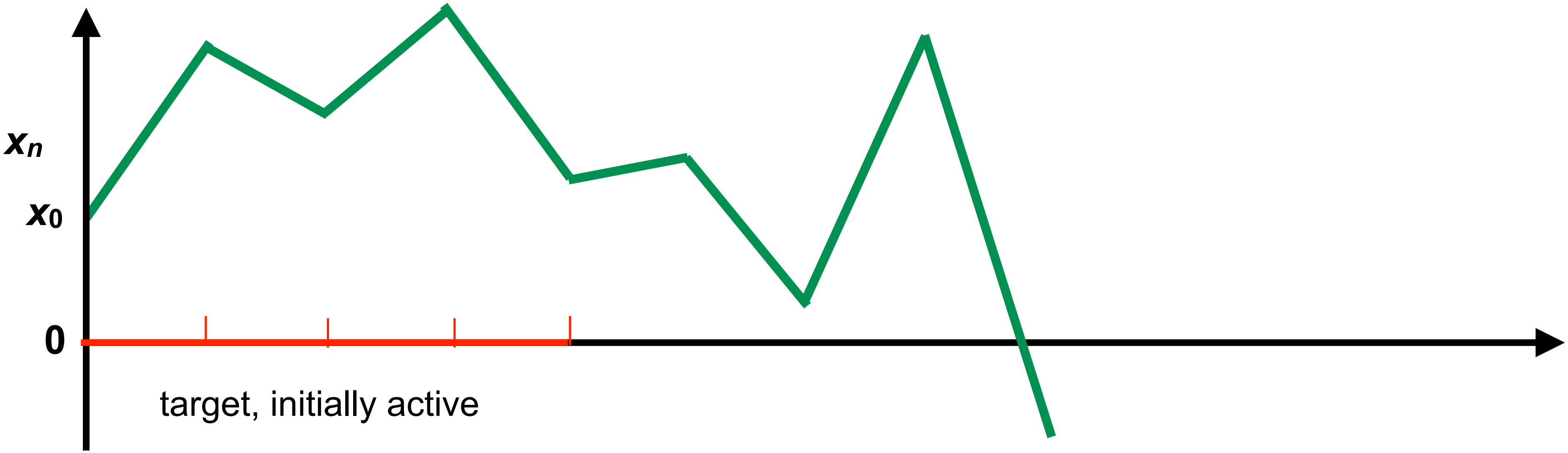
Formulation of a 1*d* model in discrete time:

$$x_0 > 0$$

$$x_n = x_{n-1} + \eta_n$$

η_n's are i.i.d. variables distributed with $f(\eta)$

$f(\eta)$ is (i) continuous (ii) symmetric: $f(-\eta) = f(\eta)$



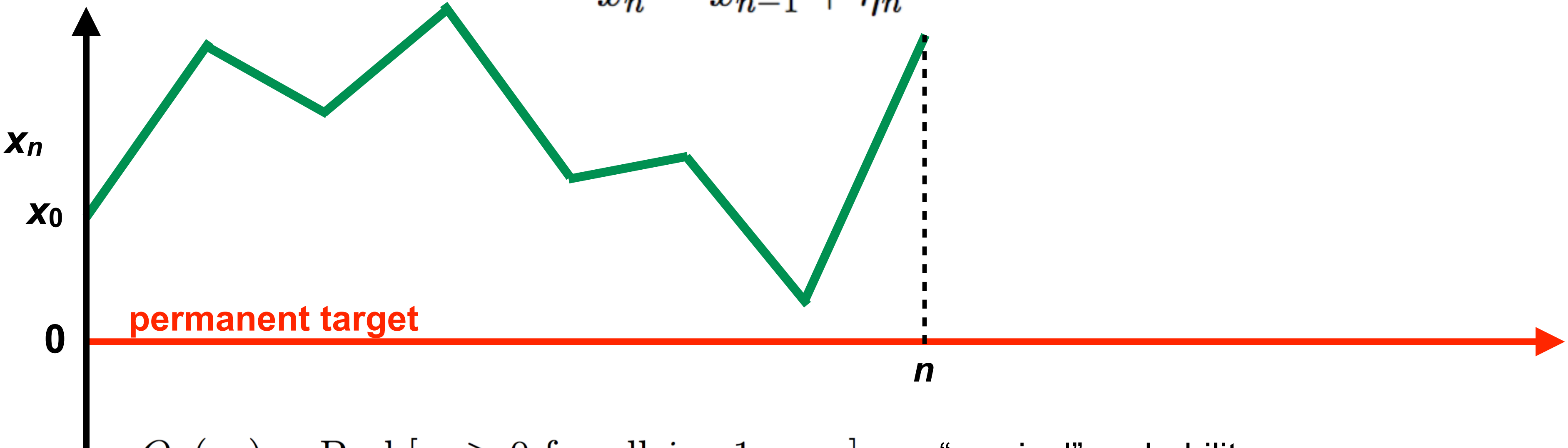
At each time step, the target stays alive with probability a,
dies with probability $1 - a$. $\langle t_{life} \rangle = 1/(1-a)$

(Markov processes for the searcher and the target)

A well known related problem (a=1):

$$x_0 > 0$$

$$x_n = x_{n-1} + \eta_n$$



$$Q_n(x_0) \equiv \text{Prob}[x_i \geq 0 \text{ for all } i = 1, ..., n]$$ "survival" probability

$$\tilde{Q}(x_0, s) \equiv \sum_{n=0}^{\infty} s^n Q_n(x_0)$$ generating function

$$\int_0^\infty \tilde{Q}(x_0, s) e^{-\lambda x_0} dx_0 = \frac{1}{\lambda\sqrt{1-s}} \exp\left[-\frac{\lambda}{\pi}\int_0^\infty \frac{\ln[1 - s\hat{f}(k)]}{\lambda^2 + k^2} dk\right]$$ Laplace transform

Pollaczek-Spitzer formula (1952-1956)

Case *a*=1:

- The survival probability decays to 0 at large *n*:

$$Q_n(x_0) \simeq \frac{U(x_0)}{\sqrt{n}}$$

The capture probability is 1.

(Majumdar, Mounaix & Schehr, J. Phys. A, 2017)

- But the mean first passage time (MFPT) is infinite:

$$T(x_0) = \sum_{n=1}^{\infty} n \left[Q_{n-1}(x_0) - Q_n(x_0)\right] = \sum_{n=0}^{\infty} Q_n(x_0) = \infty$$

Basic quantities with a finite-lived target

Capture probability:

$$C_{x_0}(a) = \sum_{n=1}^{\infty} a^{n-1}\left[Q_{n-1}(x_0) - Q_n(x_0)\right]$$

$$= \frac{1-(1-a)\widetilde{Q}(x_0, s=a)}{a}$$

maximum of $C_{x_0}(a)$ $\Longleftrightarrow$ minimum of $\widetilde{Q}(x_0, s=a)$

Conditional mean first passage time (CMFPT):

$$T_{x_0}(a) = \sum_{n=1}^{\infty} n a^{n-1}\left[Q_{n-1}(x_0) - Q_n(x_0)\right]/C_{x_0}(a)$$

$$= a\frac{\partial}{\partial a}\ln\left[1-(1-a)\widetilde{Q}(x_0, s=a)\right]$$

Basic quantities with a finite-lived target

Capture probability:

$$C_{x_0}(a) = \sum_{n=1}^{\infty} a^{n-1} \left[Q_{n-1}(x_0) - Q_n(x_0)\right]$$

$$= \frac{1 - (1-a)\widetilde{Q}(x_0, s=a)}{a}$$

maximum of $C_{x_0}(a) \iff$ minimum of $\widetilde{Q}(x_0, s=a)$

Conditional mean first passage time (CMFPT):

$$T_{x_0}(a) = \sum_{n=1}^{\infty} n a^{n-1} \left[Q_{n-1}(x_0) - Q_n(x_0)\right] / C_{x_0}(a)$$

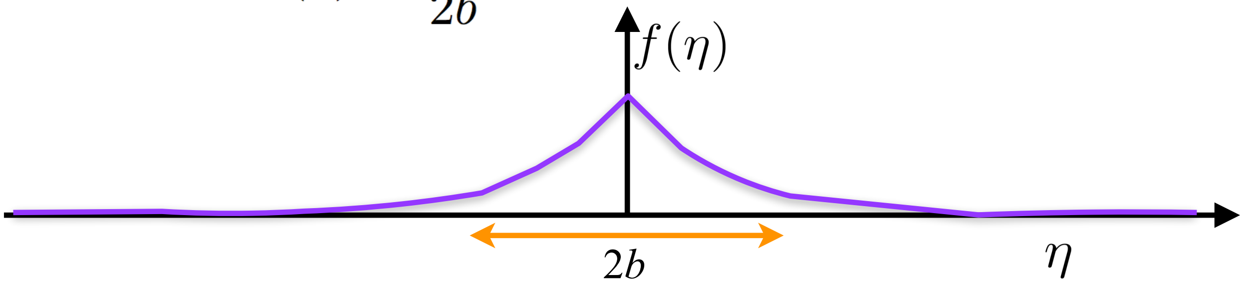
$$= a \frac{\partial}{\partial a} \ln \left[1 - (1-a)\widetilde{Q}(x_0, s=a)\right]$$

we don't have this directly

Exponential step distribution

(exactly solvable case)

$$f(\eta) = \frac{1}{2b} e^{-|\eta|/b}$$



Capture probability:

$$C_{x_0}(a) = \frac{1}{1+\sqrt{1-a}} e^{-\frac{\sqrt{1-a}}{b} x_0}$$

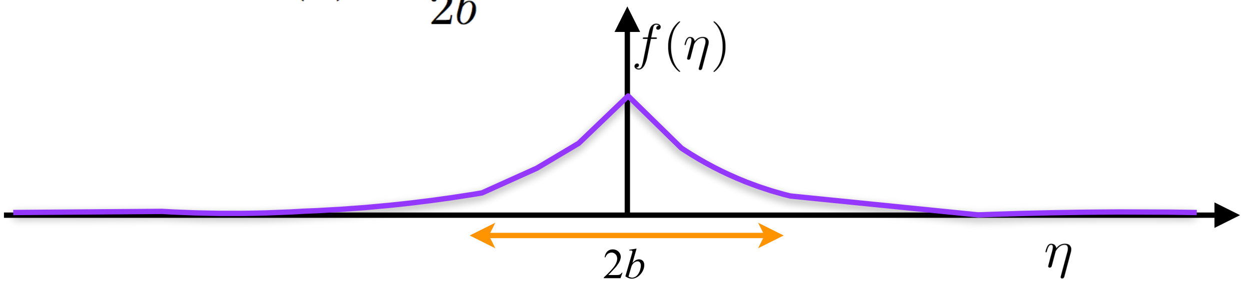
Conditional mean first passage time:

$$T_{x_0}(a) = \frac{1+\sqrt{1-a}}{2\sqrt{1-a}} + \left(\frac{a}{2\sqrt{1-a}}\right) \frac{x_0}{b}$$

Exponential step distribution

(exactly solvable case)

$$f(\eta) = \frac{1}{2b} e^{-|\eta|/b}$$



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Conditional mean first passage time:

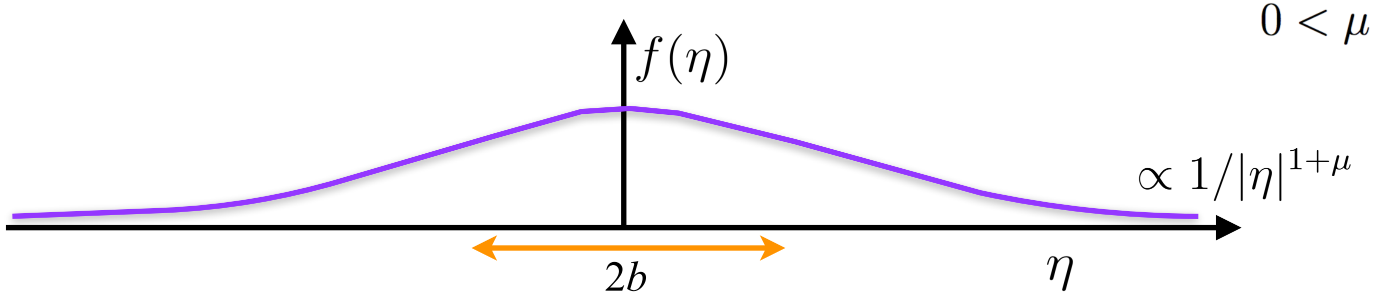
$$T_{x_0}(a) = \frac{1+\sqrt{1-a}}{2\sqrt{1-a}} + \left(\frac{a}{2\sqrt{1-a}} \right) \frac{x_0}{b}$$

Optimal parameter: $b = \infty$

Lévy step distribution

Let us consider a Lévy distribution for the RW steps: $\hat{f}(k) = e^{-|bk|^{\mu}}$

$b = 1$
$0 < \mu < 2$



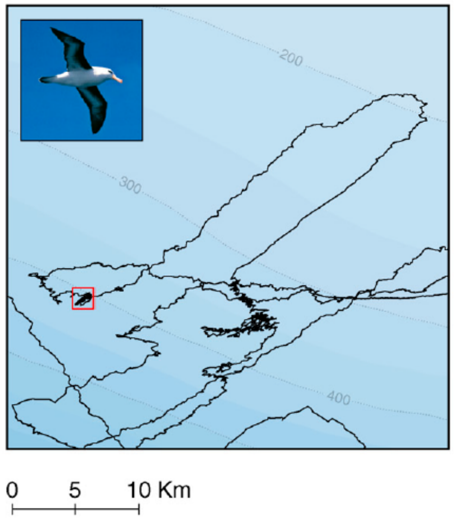
Value of μ that maximises the capture probability, or minimizes $\tilde{Q}(x_0, s = a)$: $\mu^{\star}_{cap}(x_0, a)$

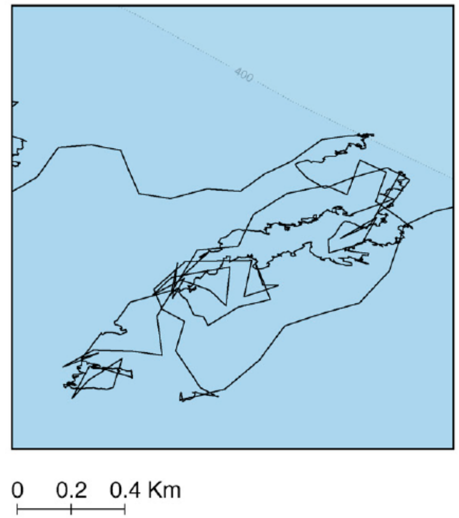
Value of μ that minimises the CMFPT: $\mu^{\star}_{FP}(x_0, a)$

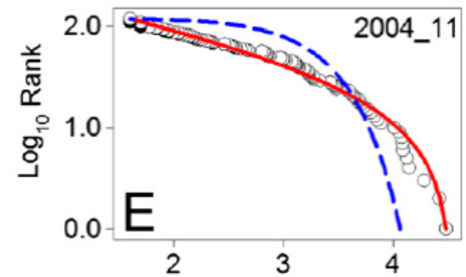
$(\mu^{\star}_{FP} \neq \mu^{\star}_{cap})$

Lévy processes in biology

Albatross



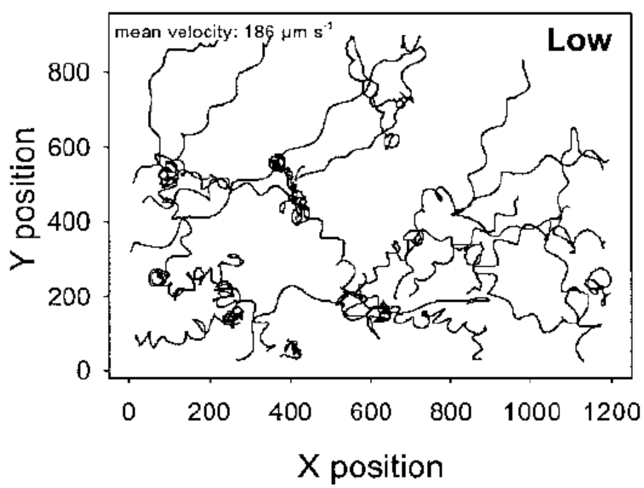




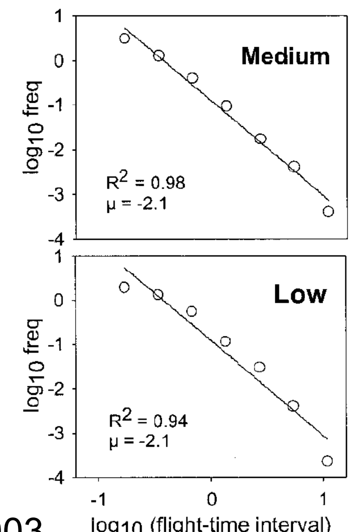
Humphries *et al. PNAS,* 2012

Microzooplankton

And: fruit flies,
mussels,
fishermen,
nomadic tribes,
bank notes,
cell phone users,
spider monkeys,
bumblebees, T-cells…



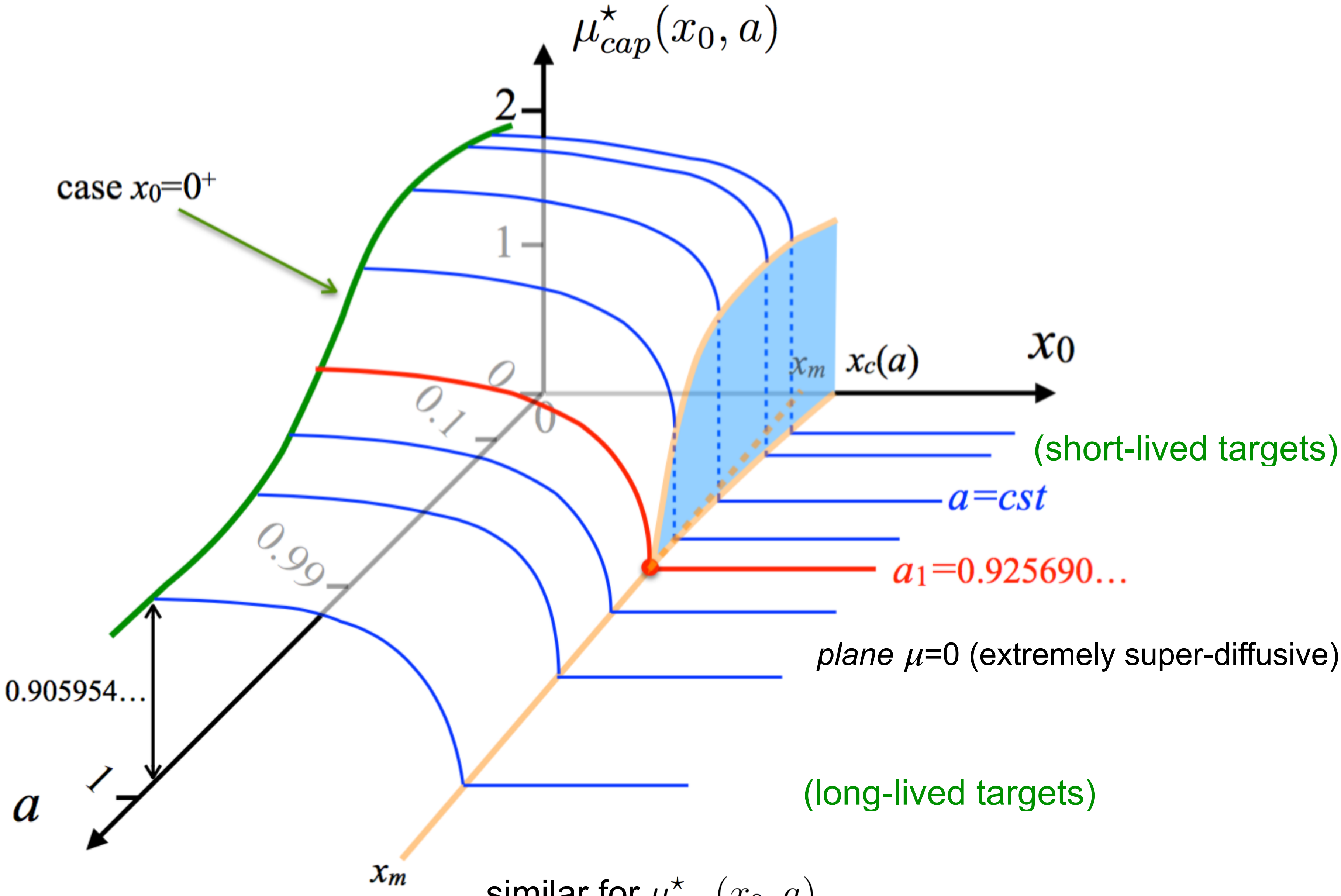
Bartumeus *et al.*, PNAS 2003



$$\mu \sim 1$$

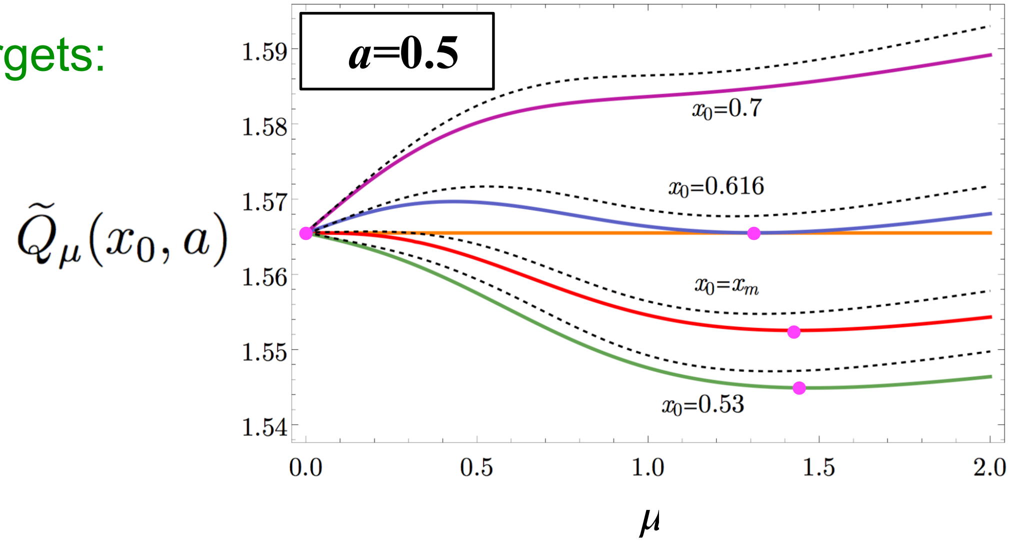
(optimal parameter in LFH, Viswanathan et al. Nature 1999.)

Main results (numerical & analytical)



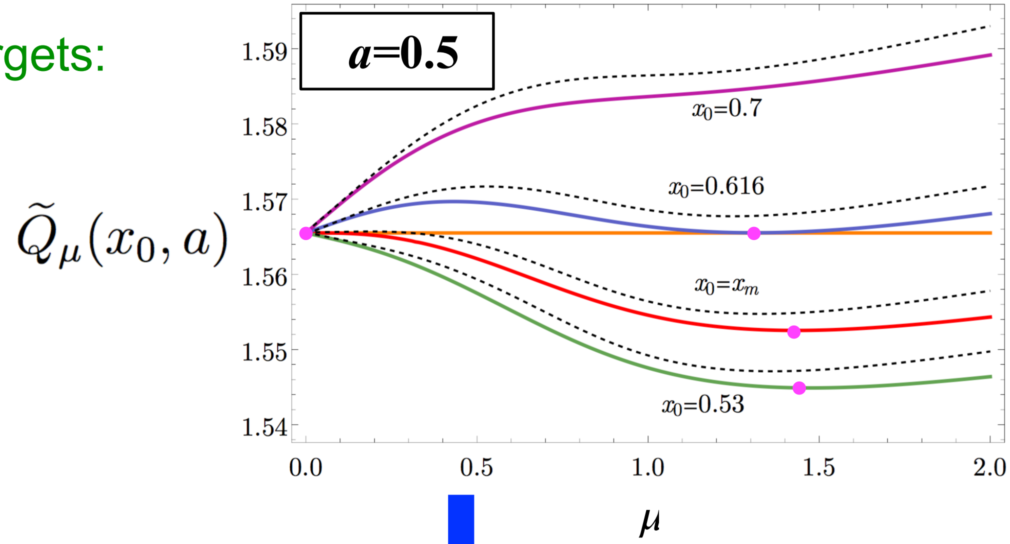
similar for $\mu^{\star}_{FP}(x_0, a)$

Short-lived targets:



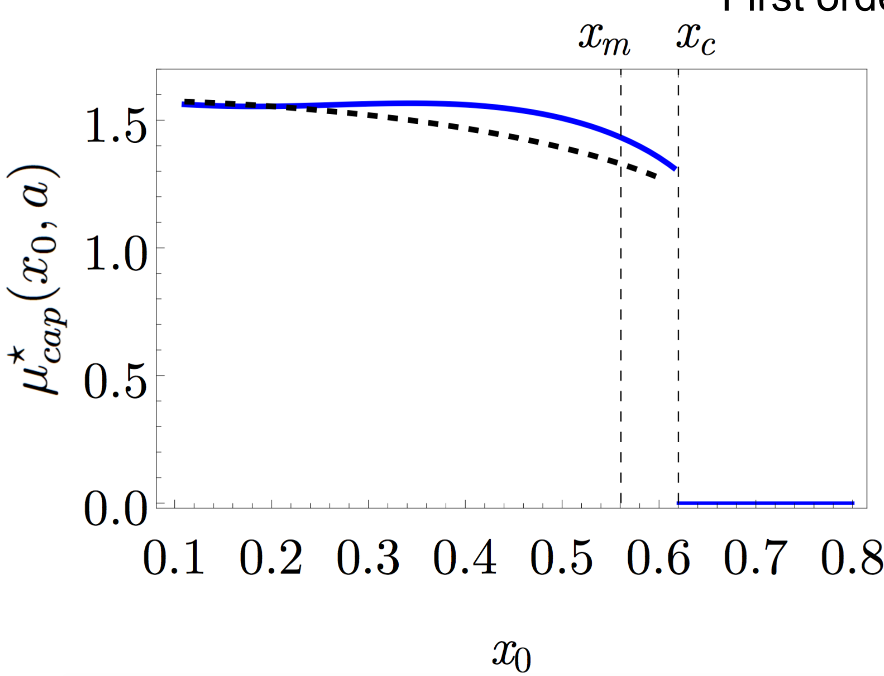
$$x_m = e^{-\gamma_E} = 0.56145...$$

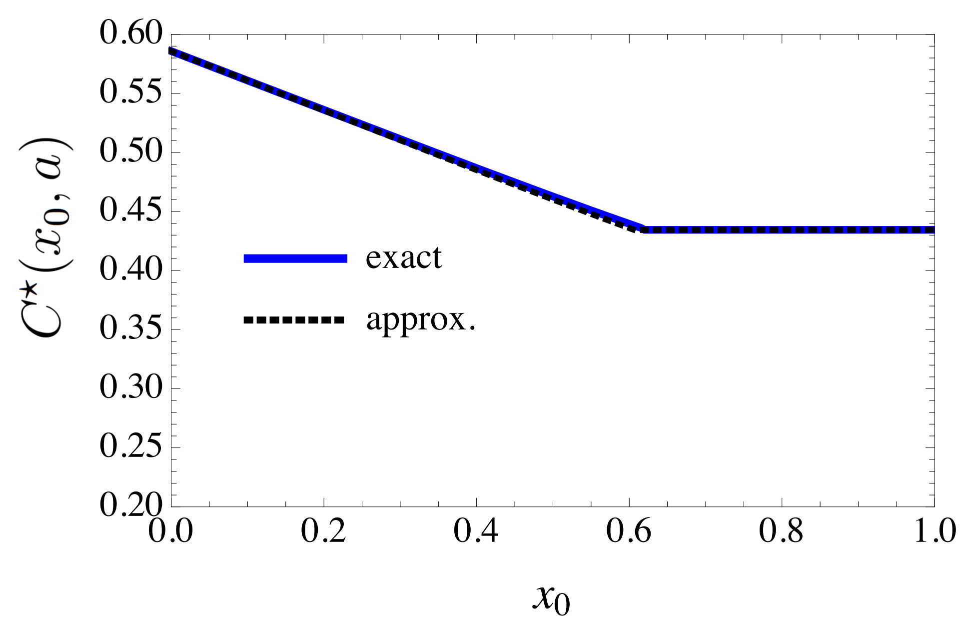
Short-lived targets:



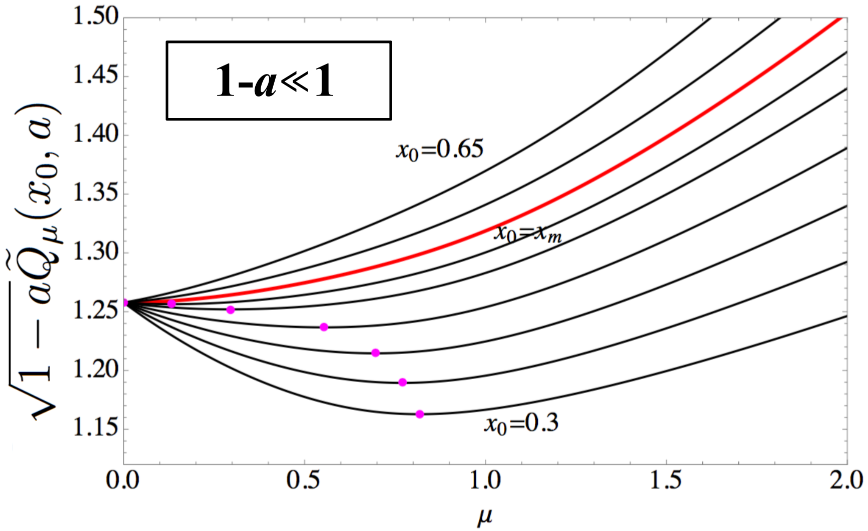
$$x_m = e^{-\gamma_E} = 0.56145...$$

First order “phase” transition





Target with a sufficiently long lifetime:

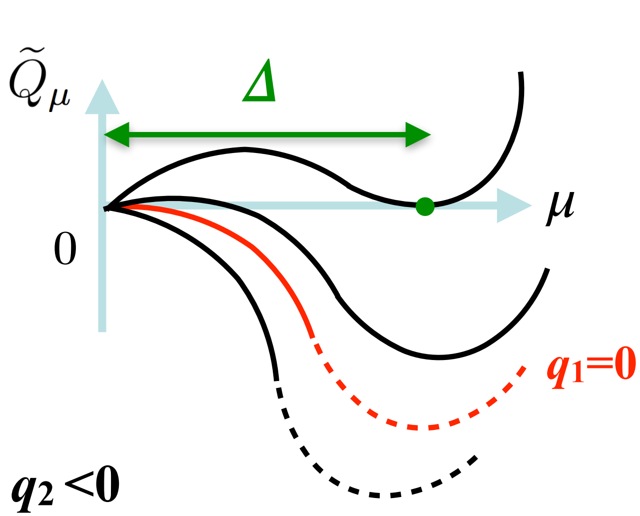


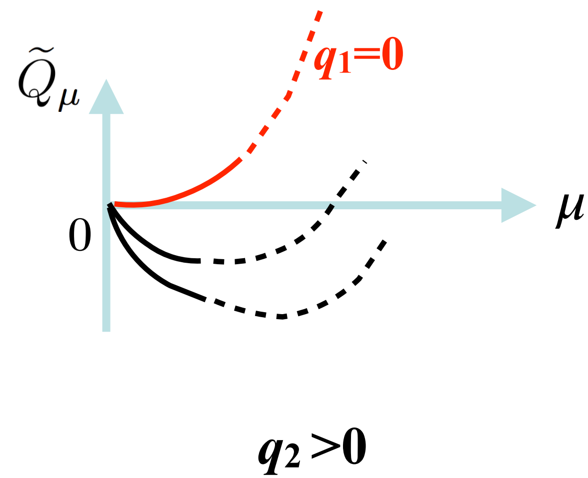
Second order “phase” transition

Series expansion at small μ:

$$\tilde{Q}_\mu(x_0, a) = q_0 + q_1\mu + \frac{1}{2!}q_2\mu^2 + \frac{1}{3!}q_3\mu^3 + \ldots$$

μ : positive "order" parameter
x_0 : "control" parameter
q_1 changes sign.





Simple scenario of first order transition: $q_2 < 0$, $q_3 > 0$.

$$\Delta = -\left.\frac{3q_2}{2q_3}\right|_{x_c}$$

Tri-critical point: $q_2 = 0$ (and $q_1 = 0$)

Non-conventional scenario of first order transition: $\boldsymbol{q_2 \geq 0,\ q_3 < 0,\ (q_4 > 0)}$

$$\widetilde{Q}(x_0, a) = q_0 + q_1\mu + \frac{1}{2!}q_2\mu^2 + \frac{1}{3!}q_3\mu^3 + \frac{1}{4!}q_4\mu^4 + \ldots$$

$$\Delta = \frac{2}{3\,q_4}\left(2\,|q_3| + \sqrt{4\,q_3^2 - 9\,q_2 q_4}\right)\Big|_{x_c}$$

Tri-critical point: $\boldsymbol{q_2 = 0,\ q_3 = 0\ \ (\text{and } q_1 = 0)}$

Non-conventional scenario of first order transition: $q_2 \geq 0$, $q_3 < 0$, $(q_4 > 0)$

$$\tilde{Q}(x_0, a) = q_0 + q_1 \mu + \frac{1}{2!} q_2 \mu^2 + \frac{1}{3!} q_3 \mu^3 + \frac{1}{4!} q_4 \mu^4 + \ldots$$

$$\Delta = \frac{2}{3\,q_4} \left(2\,|q_3| + \sqrt{4\,q_3^2 - 9\,q_2 q_4} \right) \Big|_{x_c}$$

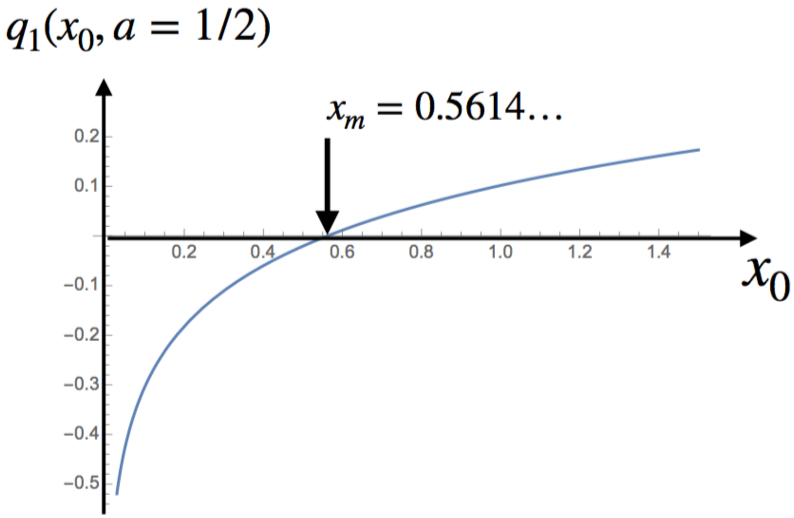
Tri-critical point: $q_2 = 0$, $q_3 = 0$ (and $q_1 = 0$)

In our problem it turns out that q_1 and q_2 always vanish at the same time (at $x_0 = x_m$)

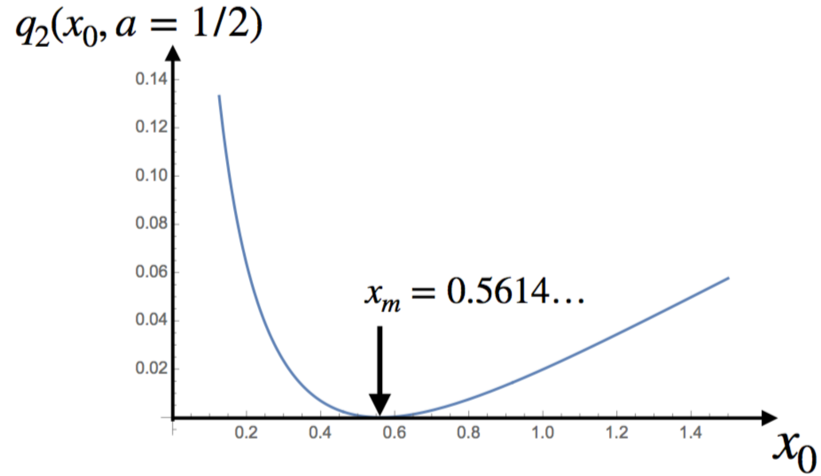
PS $\Longrightarrow$

$$q_1 = \frac{a e^{-1}}{2\sqrt{1-a}(1-ae^{-1})^{3/2}} (\ln x_0 + \gamma_E) \qquad \Rightarrow \qquad q_1 = 0 \text{ at } x_m = e^{-\gamma_E} \; \forall a$$

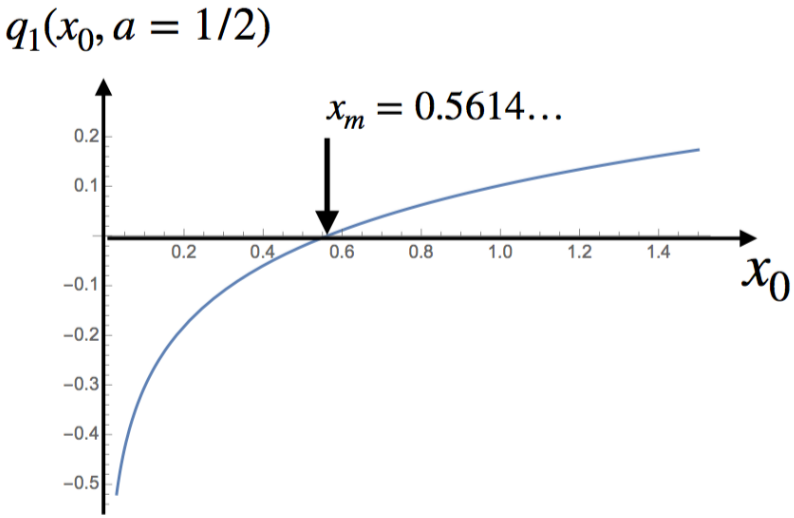
$$q_2 = \frac{3\sqrt{e} a^2}{4\sqrt{1-a}(e-a)^{5/2}} (\ln x_0 + \gamma_E)^2 \qquad \Rightarrow \qquad q_2 = 0 \text{ at } x_m = e^{-\gamma_E} \; \forall a$$

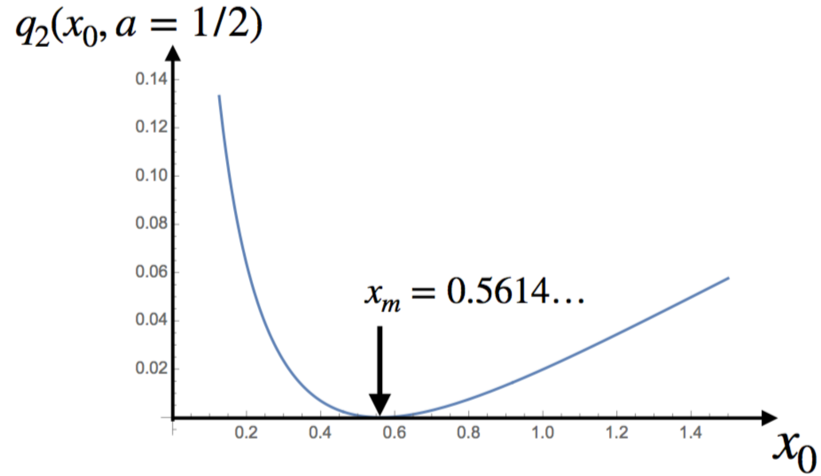


$q_1(x_0, a = 1/2)$
$x_m = 0.5614\ldots$
0.2
0.1
−0.1
−0.2
−0.3
−0.4
−0.5
0.2
0.4
0.6
0.8
1.0
1.2
1.4
x_0

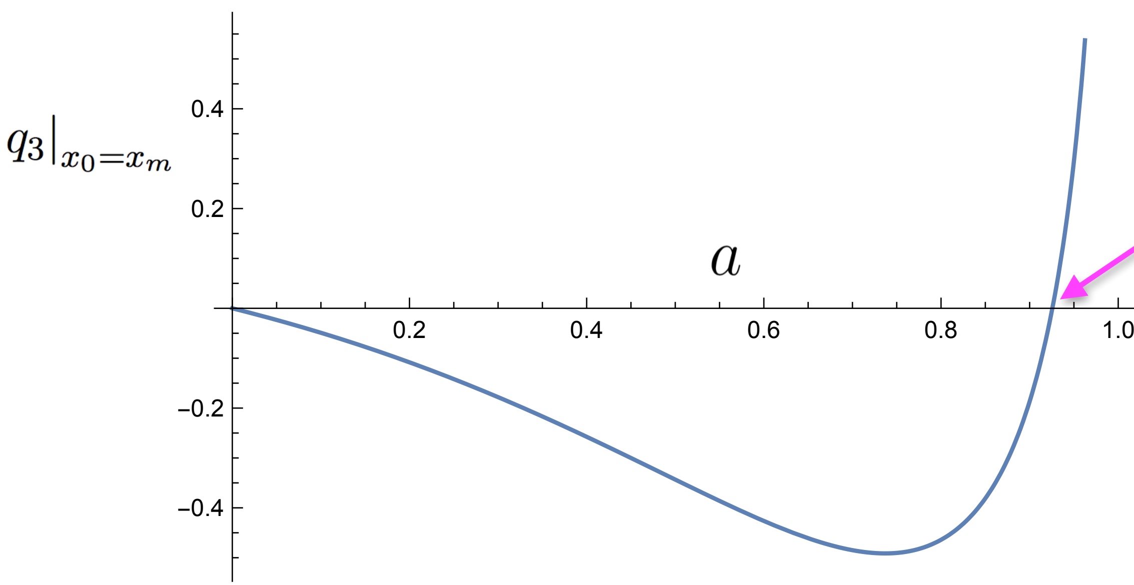


$q_2(x_0, a = 1/2)$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
$x_m = 0.5614\ldots$
0.2
0.4
0.6
0.8
1.0
1.2
1.4
x_0





$$q_3|_{x_0=x_m} = \frac{a\sqrt{e}K}{8\sqrt{1-a}(e-a)^{7/2}}(11a^2 + 8ea - 4e^2) \quad \text{with} \quad K = 2\zeta(3) = 2.4041138\ldots$$



changes sign at

$$a_1 = \frac{2e(\sqrt{15}-2)}{11} = 0.925690\ldots$$

Conditional mean first passage time

series expansion at small μ:

$$T_\mu(x_0, a) = t_0 + t_1\mu + \frac{1}{2!}t_2\mu^2 + \frac{1}{3!}t_3\mu^3 + ...$$

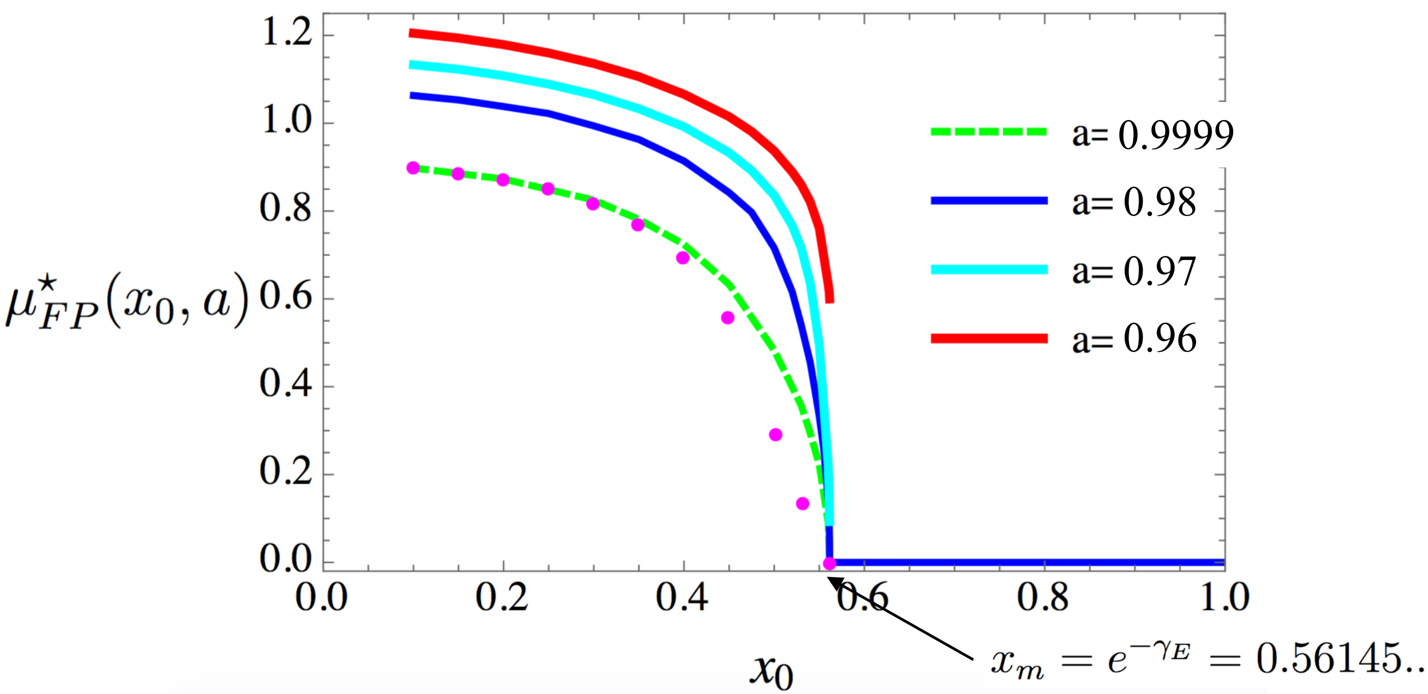
$t_3\big|_{x_0=x_m}(a)$ changes sign at $a_2 = 0.973989...$

Conditional mean first passage time

series expansion at small μ:

$$T_\mu(x_0, a) = t_0 + t_1\mu + \frac{1}{2!}t_2\mu^2 + \frac{1}{3!}t_3\mu^3 + \ldots$$

$t_3|_{x_0=x_m}(a)$ changes sign at $a_2 = 0.973989\ldots$



Approximate inversion of Pollaczek-Spitzer

Concavity of the logarithm:

$$\lambda \int_0^\infty \ln[\widetilde{Q}_\mu(x_0,s)] e^{-\lambda x_0} dx_0 \le \ln\left[\lambda \int_0^\infty \widetilde{Q}_\mu(x_0,s) e^{-\lambda x_0} dx_0\right]$$

$$\Downarrow$$

$$=$$

$$\Rightarrow \qquad \widetilde{Q}_{\mu,approx}(x_0,s) = \frac{1}{\sqrt{1-s}} \exp\left[-\frac{1}{\pi}\int_0^\infty \ln[1-s\hat{f}(k)] \frac{\sin(kx_0)}{k} dk\right]$$

- coincides with the exact $\widetilde{Q}_\mu$ in the small x_0 expansion up to $O(x_0)$,
- coincides with the exact $\widetilde{Q}_\mu$ in the small μ expansion up to $O(\mu)$.
- captures the first order transition at short lifetimes,
- captures the second order transition at long lifetimes (x_m).

Non-trivial optimal exponent for finding long-lived targets

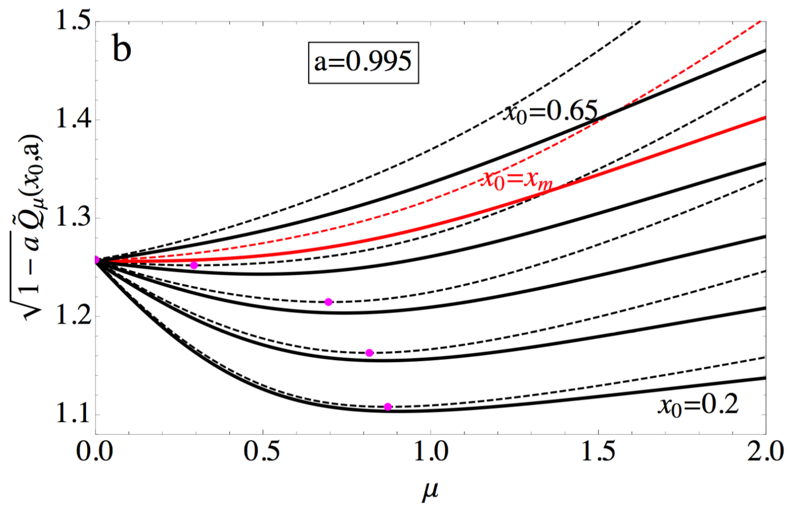
$a = 1$: mean first passage time (MFPT) is infinite $\forall f(\eta)$; all strategies are "bad".

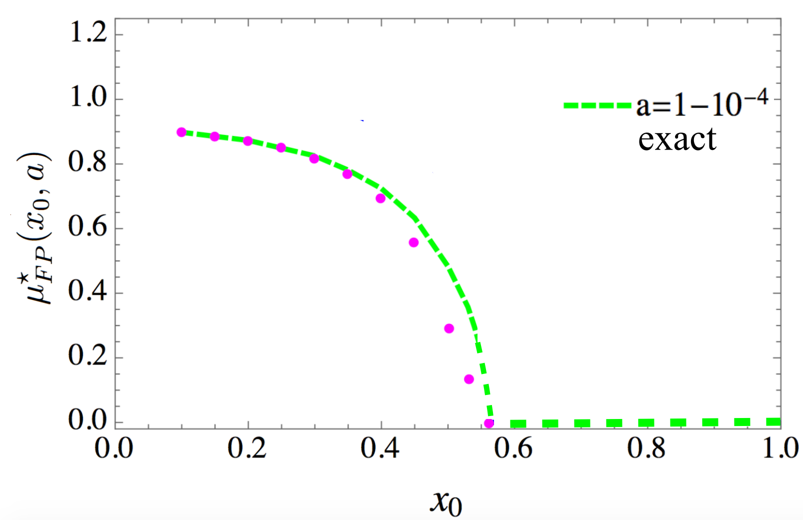
The divergence of the MFPT is due to few trajectories that take a very long time to cross the origin.

$a = 1-\varepsilon$: capture probability is nearly one; the CMFPT is large but finite, and can be optimized.

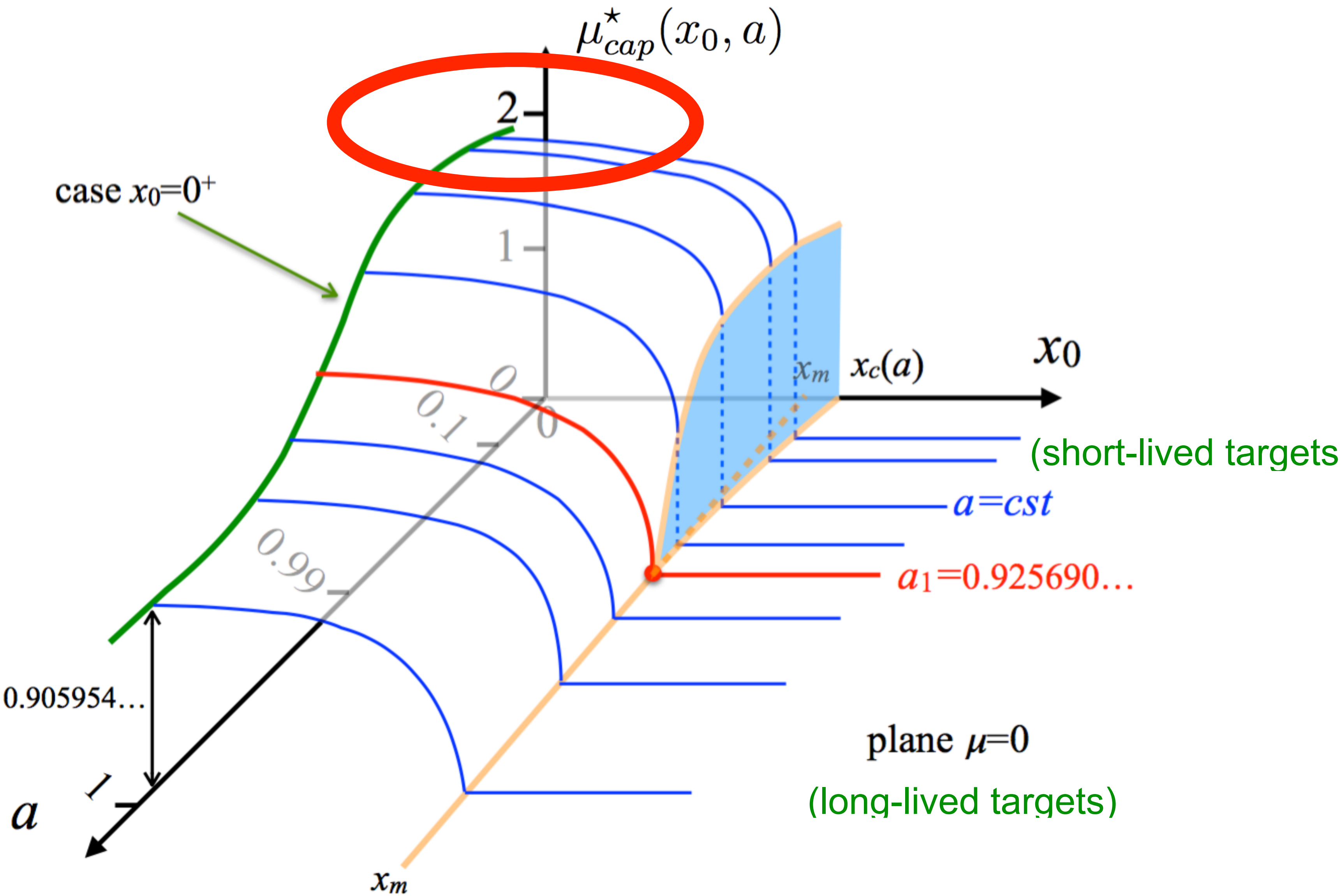
$$T_\mu(x_0, a) \simeq \frac{1}{2}\tilde{Q}_\mu(x_0, a) \qquad (\mu^\star_{FP} \simeq \mu^\star_{cap})$$

$$\simeq \frac{1}{2\sqrt{1-a}} e^{-\frac{1}{\pi}\int_0^\infty \ln[1-e^{-\left(\frac{u}{x_0}\right)^\mu}]\frac{\sin u}{u}du} \qquad \text{(concav. approx)}$$





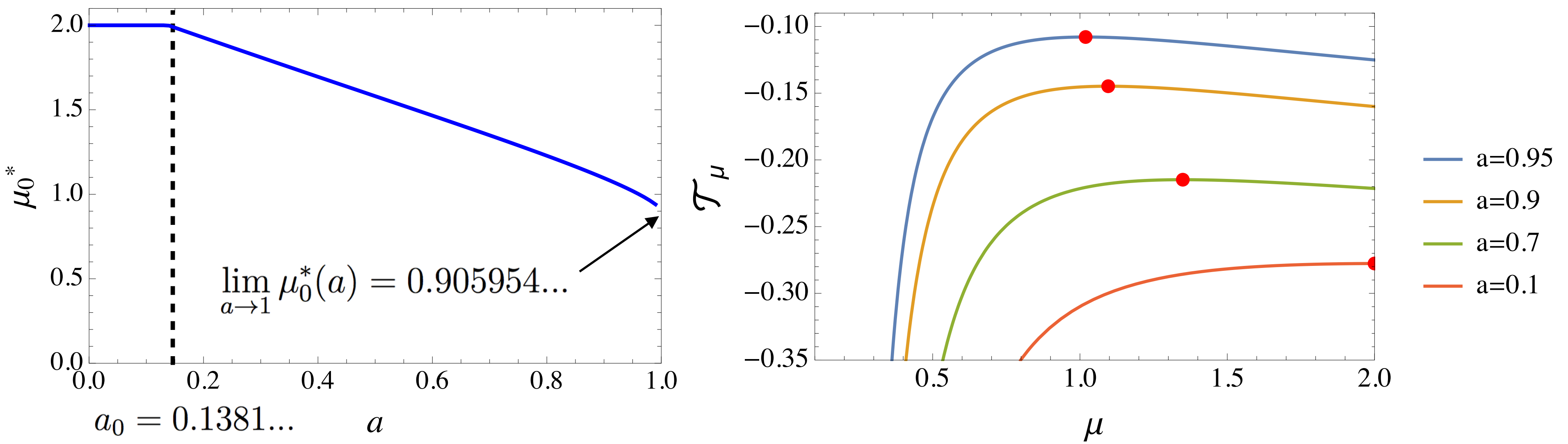
Other results



Starting very close to the target: $\lim_{x_0 \to 0^+} C_\mu(x_0, a) = \frac{1}{1+\sqrt{1-a}}$ (universal)

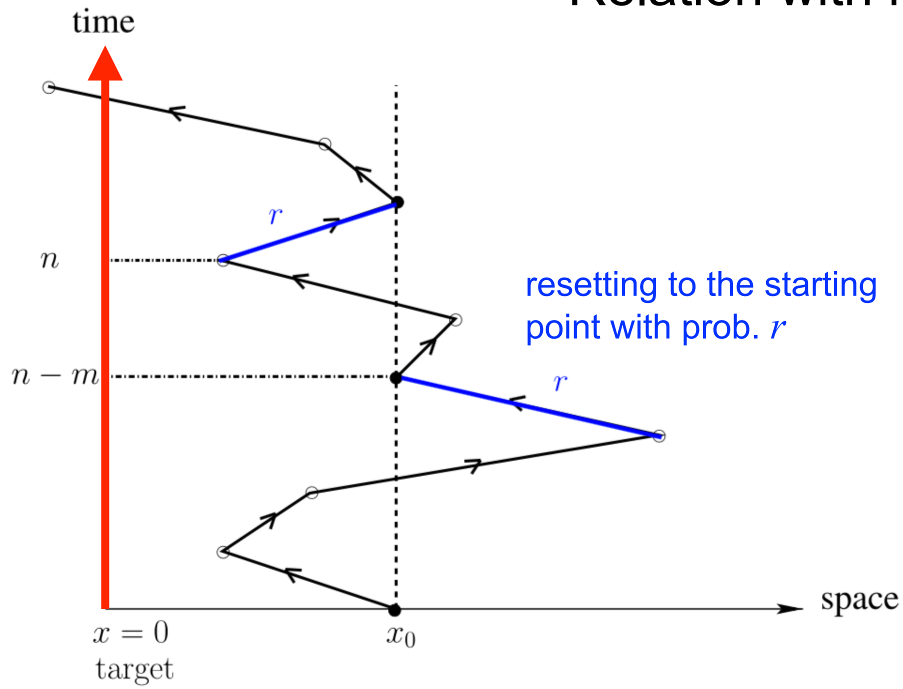
Starting close to the target: $C_\mu(x_0, a) \simeq \frac{1}{1+\sqrt{1-a}} + x_0 \mathcal{T}_\mu(a) + \mathcal{O}(x_0^2)$

$$\mathcal{T}_\mu(a) = \frac{\sqrt{1-a}}{a\pi} \int_0^\infty \ln\left[1 - a e^{-|k|^\mu}\right] dk$$



“Advise”: If you start close to the target, move as Lévy if it is long-lived but move Gaussianly if it is short-lived.

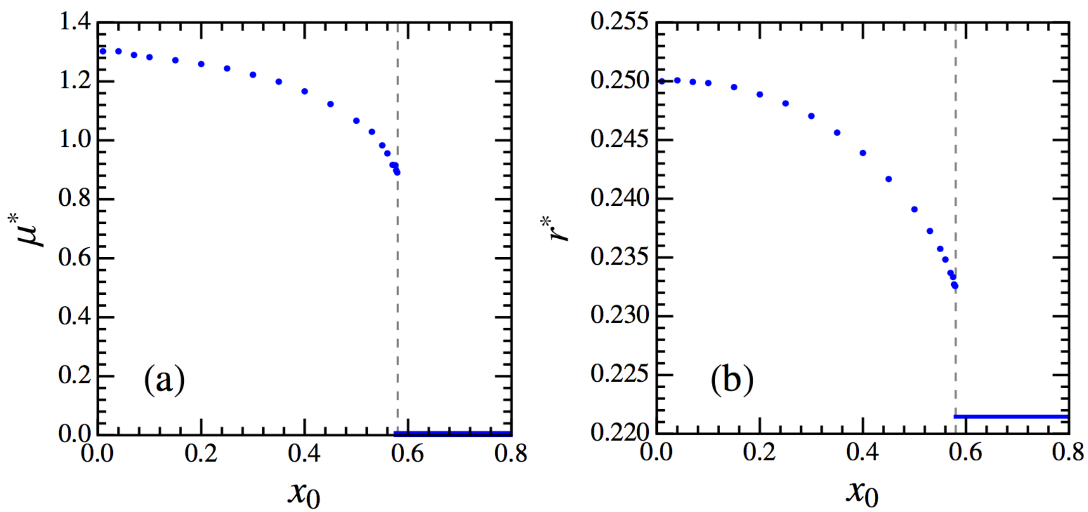
Relation with resetting processes



Mean first passage time:

$$T_{\mu}^{(resetting)}(x_0, r) = \frac{\widetilde{Q}_{\mu}(x_0, 1-r)}{1 - r\widetilde{Q}_{\mu}(x_0, 1-r)}$$

(Kusmierz, Majumdar, Sabhapandit & Schehr, PRL, 2014)



Optimization w.r. to two parameters! r and μ.

Conclusions

- Lévy flights can optimise the success of random searches with finite lifetime.
- The exponents are rather non-trivial for close-by targets and depend on the initial distance.
- Abrupt transitions for the optimal parameters, non-conventional tri-critical point.
- The infinite lifetime limit has a non trivial optimal search strategy.
- If exponential distribution of steps: “trivial” optimal distribution ($<\eta>\to\infty$).
- Lévy searches can be advantageous in uncertain environments.
- Higher dimensions?
- Other target dynamics?
- Multiple searchers?

Thanks!

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CF2019/10872