BANGALORE SCHOOL ON STATISTICAL PHYSICS XVII

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OUT-OF-EQUILIBRIUM IN HIGH DIMENSION

non reciprocity, chaos, fixed points statistics

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If you find typos please let me know!

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INTRO: SETTING, QUESTIONS, PLAN.

The setting: high-D complex ($\Rightarrow$ random) systems

(a) many components

$\vec{x} = (x_1, \dots, x_N)$ configuration

$\vec{x} \in \mathcal{C}_N$ = configuration space, $N \gg 1$

We are interested in dynamics of these systems.

Dynamics: $\frac{dx_i(t)}{dt} = F_i(\vec{x} \,|\, \hat{J}) + \varepsilon \cdot h_i(t)$

force: depends on all $\vec{x}$ because of interactions $\hat{J}$ coupling the x_i

$h_i(t)$ a field, or noise: $h_i(t) \to \eta_i(t)$ stochastic:

$\langle \eta_i(t) \rangle = 0$

$\langle \eta_i(t)\, \eta_j(t') \rangle = \delta_{ij}\, \delta(t-t')$

"white noise"

$\langle \cdot \rangle$ = average over noise

Initial condition $\vec{x}(t=0) = \vec{x}_0$ (usually "typical" $\Leftrightarrow$ uniformly distributed in $\mathcal{C}_N$) $\leftarrow$ usually called dynamics from "random" initial conditions

(b) All-to-all, heterogeneus interactions $\{J_{ij}\}_{i,j=1}^{N}$

Model them with random variables with $\mathbb{P}(\{J_{ij}\})$.

One model $\Longleftrightarrow$ many "realizations"

System with "Quenched" disorder: system evolves, but J_{ij} are fixed. [no feedback, learning, evolution]

Want to make statistical statements, e.g. "what is the behaviour of most of the realizations when $N \gg 1$?".

Notation: $\mathbb{E}[\cdot]$, $\mathbb{P}(\cdot)$ are w.r.t. random couplings

(c) The interactions can be non-reciprocal: $J_{ij} \neq J_{ji}$.

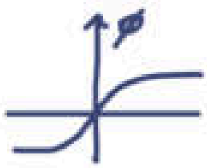
Means that in general, $F_i(\vec{x}) \neq -\partial/\partial x_i \, \mathcal{E}(\vec{x})$ with $\mathcal{E}$ = energy, potential...

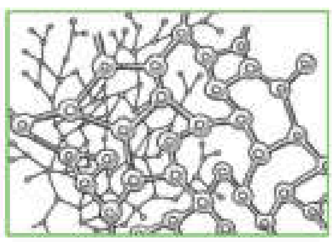
■ Two examples from complex systems theory

(i) BRAINS – Biological neural networks

x_i = membrane potential of neuron i

$\vec{x} \in \mathcal{C}_N = \mathbb{R}^N$





$$\frac{dx_i(t)}{dt} = \underbrace{-\mu x_i(t)}_{\text{one-body term}} + \frac{\sigma}{\sqrt{N}} \underbrace{\sum_{j\neq i} J_{ij}\, \phi\big(x_j(t)\big)}_{\text{inputs from all other neurons (non-linear!)}}$$

[Wants $x_i(t) \to 0$]

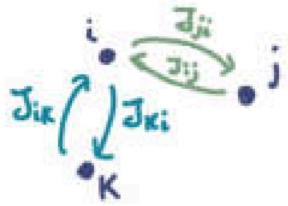
J_{ij} = synaptic weight

Interactions J_{ij} (synaptic connections): random Gaussian
All independent, except J_{ij} and J_{ji}.

Set: $\mathbb{E}[J_{ij}] = 0$,

degree of reciprocity

$$\mathbb{E}[J_{\ell m} J_{kn}] = \delta_{\ell k}\delta_{mn} + \alpha\, \delta_{\ell n}\delta_{mk}$$



$\alpha = 1 \Rightarrow J_{ij} = J_{ji}$

$\alpha = 0 \Rightarrow J_{ij}$ and J_{ji} are independent.

Scaling with N: $\sum_{j=1}^{N} J_{ij}\, \phi(x_j) \sim \mathcal{O}(\sqrt{N})$ when $N \gg 1$!

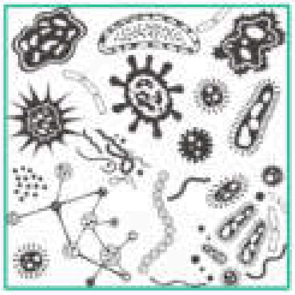
Seminal works: CRISANTI, SOMPOLINSKY 1987
SOMPOLINSKY, CRISANTI, SOMMERS 1988

A lot of theory work up to these days. Examples:
CLARK, ABBOTT 2024 STEINBERG, ADVANI, SOMPOLINSKY 2021
BERLEMONT, MONGILLO 2022 [...] MANY OTHERS

(ii) ECOSYSTEMS - Population dynamics

x_i = abundance of species i

$x_i \geq 0$, $\vec{x} \in \mathcal{C}_N = \mathbb{R}_+^N$



$$\frac{dx_i(t)}{dt} = \underbrace{x_i(t)\left[\mu_0 - x_i(t)\right.}_{\text{One body term}} - \frac{\mu}{N}\sum_{j=1}^{N} x_j(t) \underbrace{- \frac{\sigma}{\sqrt{N}}\sum_{j=1}^{N} J_{ij}\, x_j(t)\Big]}_{\text{interactions}}$$

One body term
[wants $x_i \to \dfrac{\mu_0}{1 + A e^{-\mu_0 t}}$]

interactions
non-linear forces!

Seminal works: RIEGER 1989
OPPER, DIEDERICH 1989

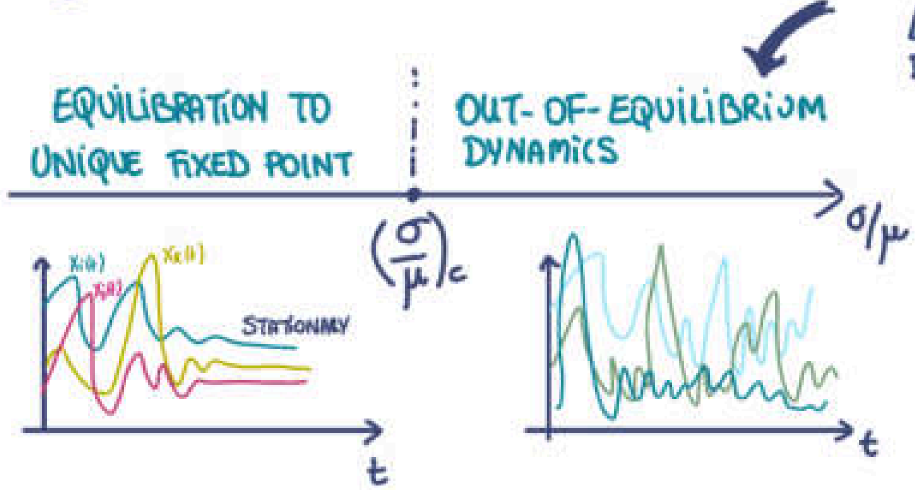
A "revival" of these high-D models in recent days.

Examples: PEARCE, AGARWALA, FISHER 2020
GALLA 2018
ROY, BIROLI, BUNIN, CAMMAROTA 2019
ARNOULX DE PIREY, BUNIN 2024
[...] MANY OTHERS

▨ The phenomenology of these models (when $N \to \infty$)

Dynamics from "random" initial conditions.

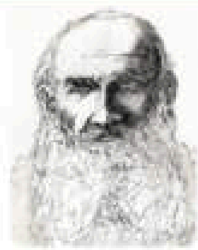
? we are interested in this regime



<u>How</u> is system out-of-equilibrium?

In some sense: ALL EQUILIBRATING SYSTEMS ARE ALIKE.* EACH NON-EQUILIBRATING SYSTEM IS NON-EQUILIBRATING IN ITS OWN WAY.

* T.T.I. = time-translation invariance
FDT = fluctuation-dissipation relations
G-B = Gibbs-Boltzmann measures



L. Tolstoj

Three Questions:

Q1. Which kind of out-of-equilibrium? We will see two examples: AGING, CHAOS.

Q2. How much non-reciprocity changes the dynamics?

Q3. Studying dynamics is complicated! Can one understand it from "statics"?

Example: if system equilibrates at temperature $T = 1/\beta$, its large-time behaviour understandable from Gibbs-Boltzmann static measure:

$$\mu_\beta(\vec{x}) \sim e^{-\beta \mathcal{E}(\vec{x})} \qquad \mathcal{E}(\vec{x}) \text{. energy.}$$

Dynamics of complex, random, high-N systems is a challenge. Here, focus on simple model for which can do explicit analysis.

▨ The Plan:

I. Simple (non-reciprocal) models

II. A warm-up: low-N dynamics

CONCEPTS: relaxational dynamics, landscapes, equilibrium

TOOLS: decomposition into modes, differential equations.

III. Large N, linear random forces.

CONCEPTS: concentration, self-averagingness, separation of timescales, aging, slow (algebraic) vs fast decay, universality.

TOOLS: random matrix theory, large-N asymptotics

⟹ Mostly based on: STARIOLO, METZ, JSTAT (2026) 033301

IV. Large N, non-linear random forces

CONCEPTS: self-consistent processes, DMFT equations, rugged landscapes, fixed points vs chaos

TOOLS: DMFT, Hamiltonian dynamics, Kac-Rice formalism

⟹ Mostly based on: FOURNIER, PACCO, VR, URBANI, PRE 113 (2026) 044139

I. SIMPLE NON-RECIPROCAL MODELS

THREE MODELING INGREDIENTS:

(1) Confinement.

Examples above have a "confinement" term that contrasts $|x_i| \to \infty$.

Can confine dynamics on a compact manifold, e.g. on a sphere: "spherical constraint".

$\vec{x} \in \mathcal{C}_N = \{ \vec{x} : \|\vec{x}\|^2 = N \}$

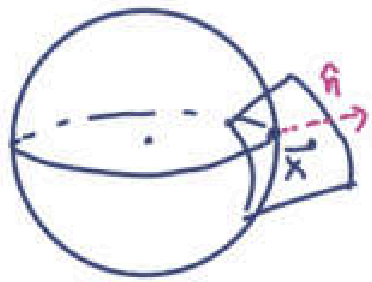
How is this implemented? Lagrange multiplier $z(\vec{x}(t))$

$$\frac{dx_i(t)}{dt} = F_i(\vec{x}(t)) = f_i(\vec{x}(t)) + z(\vec{x}(t))x_i \qquad z(t) := z(\vec{x}(t))$$

$z(t)$ choosen at each time to enforce $\vec{x}^T \cdot \vec{x} = N$:

$$\frac{d}{dt}\|\vec{x}\|^2 = 0 = 2\sum_{i=1}^N x_i(t) \cdot \frac{dx_i(t)}{dt} = \vec{f} \cdot \vec{x} + z(t) \cdot N$$

$$\Longrightarrow z(t) = -\frac{1}{N}\vec{f} \cdot \vec{x} = -1/\sqrt{N}\, \vec{f} \cdot \hat{n}$$



Geometrical interpretation

subtract to r.h.s. of dynamics the component of the force that is not tangent to sphere!

$$\hat{n} = \vec{x} / \sqrt{\|\vec{x}\|^2}$$

(2) Competing terms.

Competition in dynamics between "easy", often deterministic/uniform terms (→ dynamics with them alone is "easy") and random, non-linear terms.

[⇒ dynamical transitions when $N \to \infty$]

E.g.
$$f_i(\vec{x}) = \underbrace{\frac{\mu}{N} \sum_{j=1}^{N} x_j}_{\substack{\text{uniform} \\ \text{simple: all components see} \\ \text{same field}}} + \frac{\sigma}{N^{\frac{p-1}{2}}} \sum_{i_2, \dots, i_p} \underbrace{J_{i i_2 \dots i_p} x_{i_2} \cdots x_{i_p}}_{\substack{\text{random interactions} \\ \text{heterogeneous}}}$$

$p=2 \Rightarrow$ binary interactions $\Rightarrow$ linear forces

$p>2 \Rightarrow$ higher-order interactions $\Rightarrow$ non-linear forces

If interested in out-of-equilibrium phase only: set $\mu=0, \sigma=1$.

(3). Randomness & non-reciprocity.

- The interactions & the force have "simple" statistics: the force is a Gaussian vector.
- Non reciprocity: $J_{ijk} \neq J_{jik}, J_{kij}, \dots$

  The force can have a symmetric and an asymmetric part.

How to encode this?

<u>Modelling interactions: $p=2$</u>

Let $J_{ij} = \sqrt{1-\alpha}\; a_{ij} + \sqrt{\alpha}\; S_{ij} \quad \alpha \in [0,1]$

where $\{a_{ij}\}_{ij}$ iid Gaussian $\sim \mathcal{N}(0,1)$

and $\{S_{ij}\}_{i<j}$ iid Gaussian $\sim \mathcal{N}(0,1)$

and $S_{ij} = S_{ji}$, S_{ii} iid Gaussian $\sim \mathcal{N}(0,2)$

$$\Longrightarrow \mathbb{E}[J_{ij}^2] = (1-\alpha)\,\mathbb{E}[a_{ij}^2] + \alpha\,\mathbb{E}[S_{ij}^2] = 1 + \alpha\,\delta_{ij}$$

$$\mathbb{E}[J_{ij} J_{ji}] = \alpha\,\mathbb{E}[S_{ij}^2] = \alpha$$

In summary: $\mathbb{E}[J_{\ell m} J_{kn}] = \delta_{\ell k}\delta_{mn} + \alpha\, \delta_{\ell n}\delta_{mk}$

The J_{ij} are entries of a random matrix, taken from a random matrix ensemble called "real Elliptic ensemble". Ensemble of real matrices with Distribution:

$$\mathbb{P}_{N,\alpha}(J)\,dJ = A_{N,\alpha}\, e^{-\frac{1}{2}\frac{1}{1-\alpha^2}\left\{\mathrm{Tr}(JJ^T) - \alpha\,\mathrm{Tr}(J^2)\right\}} \prod_{i,j} dJ_{ij}$$

Notice: $\mathbb{P}_{N,\alpha}(J)\,dJ = P_{N,\alpha}(J^T)\,dJ$

EX. 1

Check that the distribution reproduces the covariances, and that

$$A^{-1}_{N,\alpha} = (2\pi)^{N^2/2}(1+\alpha)^{N/2}(1-\alpha^2)^{\frac{N(N-1)}{4}}$$

When $\alpha \to 1$, matrix is symmetric: "Gaussian orthogonal ensemble" (GOE).

$$\mathbb{P}_{N,\alpha}(J)\,dJ \xrightarrow{\alpha\to 1} a_N\, e^{-\frac{1}{4}\mathrm{Tr}\{J^2\}} \prod_{i\le j} dJ_{ij}$$

The name comes from the fact that distribution is invariant under $J \to J_R = OJO^T$ with $OO^T = \mathbb{1}$

Modelling interactions: $p > 2$ (eg. $p=3$)

The couplings J_{ijk} form a Gaussian tensor.

Can decompose: $J_{ijk} = \sqrt{1-\alpha}\, a_{ijk} + \sqrt{\alpha}\, S_{ijk}$

Where now

$$\mathbb{E}[a_{ijk} a_{lmn}] = \delta_{il}\left[\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}\right]$$ symmetric in two last indices

$$\mathbb{E}[S_{ijk} S_{lmn}] = \sum_{\pi} \delta_{i\pi(l)}\delta_{j\pi(m)}\delta_{k\pi(n)}$$ π - permutation

(these choices of decomposition are made to be consistent with Part IV of notes)

In summary, we consider here models of the form:

$$\frac{dx_i}{dt} = \frac{\mu}{N}\sum_{j=1}^{N} x_j + \frac{\sigma}{N^{\frac{p-1}{2}}}\sum_{i_2,\dots,i_p} J_{i i_2 \dots i_p} x_{i_2}\dots x_{i_p} + z(t) x_i + \varepsilon h_i(t)$$

$x_i(0) = x_i^0 \in \mathcal{C}_N = \{\vec{x} : \|\vec{x}\|^2 = N\}$

III. WARM-UP: FINITE N, LINEAR

Goals of part II:

1. define relevant observables (correlation, response...)
2. recap linear equations + asymmetric matrices
3. reciprocal interactions: equilibration, fixed points

■ Take $p=2$. $\sum_{i=1}^{N} x_i^2(t) = N$ spherical constraint

$$\frac{dx_i(t)}{dt} = \frac{\sigma}{\sqrt{N}} \sum_{j(\neq i)}^{N} J_{ij} x_j + \frac{\mu}{N} \sum_{j=1}^{N} x_j + z(t) x_i + \varepsilon h_i(t)$$

$$\xRightarrow{\text{transpose}} \quad \frac{d\vec{x}^T}{dt} = \vec{x}^T \frac{\sigma}{\sqrt{N}} J^T + \frac{\mu}{N} \vec{x}^T \left(\vec{1} \cdot \vec{1}^T\right) + \vec{x}^T z(t) + \varepsilon \vec{h}^T(t)$$

Call $A = -\left(\frac{\sigma}{\sqrt{N}} J^T + \frac{\mu}{N} \vec{1}\,\vec{1}^T\right)$ with $\vec{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

Then: $$\frac{d\,\vec{x}^T(t)}{dt} = \vec{x}^T(t) \cdot \left(-A + z\mathbb{1}\right) + \varepsilon\, \vec{h}^T(t)$$

where $\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $N \times N$

Notice: here $\vec{x}^T$ denotes row vectors.

▨ What to look at? Correlation & Response

$m(t) = 1/_N \sum_{i=1}^{N} x_i(t)$ ⇒ how uniform are configurations

$$C(t,t') = \frac{1}{N}\sum_{i=1}^{N} x_i(t)\, x_i(t')\Big|_{\varepsilon=0} = \frac{1}{N}\bar{x}^T(t)\cdot\bar{x}(t')\Big|_{\varepsilon=0}$$

⟹ How much configurations are similar at ≠ times

$$R(t,t') = \frac{1}{N}\sum_{i=1}^{N} \frac{\delta x_i(t)}{\delta[\varepsilon h_i(t')]}\Big|_{\varepsilon\to 0}$$ ⟹ How much the trajectory changes due to small kick (Linear Response)

By causality: $t' \leq t$ for response.

If system has long memory, R decays slowly.

II.A. SOLUTION OF DYNAMICS

How to proceed? Diagonalize A and work with modes (eigenvectors). However, $A \neq A^T$!

Real, Asymmetric matrices: facts.

(i) Eigenvalues λ_α are real, or complex-conjugate pairs: if λ_α is evalue, λ_α^* is too!

(ii) Right and left eigenvectors are different:

$J\vec{R}_\alpha = \lambda_\alpha \vec{R}_\alpha$, $J^T \vec{L}_\alpha = \lambda_\alpha \vec{L}_\alpha$, $(\vec{L}_\alpha)^T J = \lambda_\alpha (\vec{L}_\alpha)^T$

Notice: J and J^T have same eigenvalues.

(iii) The vectors $\{\vec{R}_\alpha\}_{\alpha=1}^{N}$ and $\{\vec{L}_\alpha\}_{\alpha=1}^{N}$ are not orthogonal: $(\vec{R}_\alpha)^T \cdot \vec{R}_\beta \neq \delta_{\alpha\beta}$

But biorthogonal: $(\vec{L}_\alpha)^T \cdot \vec{R}_\beta = \delta_{\alpha\beta}$

$\mathbb{1} = \sum_\alpha \vec{R}_\alpha \vec{L}_\alpha^T$

Notice: invariance under $\vec{R}_\alpha \to c_\alpha \vec{R}_\alpha$, $\vec{L}_\alpha \to c_\alpha^{-1} \vec{L}_\alpha$

(iv) If λ_α real, eigenvectors can be chosen real.

☒ Notation from now on:

$\vec{x}$ (column vector) $\Rightarrow |x\rangle$

$\vec{x}^T$ (row vector) $\Rightarrow \langle x|$

Scalar products: $\vec{v}^T \cdot \vec{u} = \langle v|u\rangle$

Resolution of identity: $\mathbb{1} = \sum_\alpha |R_\alpha\rangle\langle L_\alpha|$

Notice: $\langle x|$ is not transpose + conjugation. Only transpose!

- The equation reads:

$$\frac{d\langle x(t)|}{dt} = -\langle x(t)|A + z(t)\langle x(t)| + \varepsilon\langle h(t)| \qquad (*)$$

Treat $z(t)$ as a function of time, to be fixed once we know the solution for $|x(t)\rangle$.

- Assume $\{\lambda_1, \lambda_2, \dots \lambda_N\}$ eigenvalues of A with eigenvectors: $A|R_\alpha\rangle = \lambda_\alpha |R_\alpha\rangle$, $\langle L_\alpha|A = \lambda_\alpha\langle L_\alpha|$

Define the coefficients:

$$r_\alpha(t) = \vec{x}^T(t)\cdot\vec{R}_\alpha = \langle x(t)|R_\alpha\rangle$$

$$\ell_\alpha(t) = \vec{x}^T(t)\cdot\vec{L}_\alpha = \langle x(t)|L_\alpha\rangle$$

CAN BE COMPLEX!

$$h^R_\alpha(t) = \vec{h}^T(t)\cdot\vec{R}_\alpha = \langle h(t)|R_\alpha\rangle$$

$$h^L_\alpha(t) = \vec{h}^T(t)\cdot\vec{L}_\alpha = \langle h(t)|L_\alpha\rangle$$

- The correlation function:

$$C_N(t,t') = \frac{1}{N}\vec{x}^T(t)\cdot\vec{x}(t') = \frac{1}{N}\langle x(t)|\left(\sum_{\alpha=1}^{N}|R_\alpha\rangle\underbrace{\langle L_\alpha|\right)|x(t')\rangle}$$

$$= \frac{1}{N}\sum_{\alpha=1}^{N} r_\alpha(t)\,\ell_\alpha(t')$$

Notice: $\left(\vec{L}^T_\alpha\cdot\vec{x}(t')\right)^T = \vec{x}^T(t')\cdot\vec{L}_\alpha$

The response:

$$R_N(t,t') = \frac{1}{N} \lim_{\varepsilon \to 0} \sum_{i=1}^{N} \frac{\delta x_i(t)}{\delta[\varepsilon h_i(t')]}$$

$$= \frac{1}{N} \lim_{\varepsilon \to 0} \sum_{\alpha, i} \frac{\delta r_\alpha(t)}{\delta[\varepsilon h_i(t')]} \langle L_\alpha | i \rangle$$

Use that

$x_i(t) = \hat{e}_i^T \cdot \vec{x}(t)$

$\langle i | x(t) \rangle$

$|x\rangle = \sum_\alpha \langle x | R_\alpha \rangle \langle L_\alpha |$

Mode dynamics.

Sandwich (*) with $|R_\alpha\rangle$ to get:

non-homogeneous differential eq.

$$\frac{d r_\alpha(t)}{dt} = -\lambda_\alpha r_\alpha(t) + z(t) r_\alpha(t) + \varepsilon h_\alpha^R(t)$$

Solved by: $r_\alpha(t) = \frac{r_\alpha(0) e^{-\lambda_\alpha t}}{\Gamma_\varepsilon(t)} + \varepsilon \int_0^t ds\, K_\alpha(t-s) h_\alpha^R(s)$

solution of homogeneous

convolution with green function

where $\Gamma_\varepsilon(t) = e^{-\int_0^t du\, z(u)}$, $K_\alpha(u) = e^{-\lambda_\alpha u} / \Gamma_\varepsilon(u)$

Notice: $\Gamma_\varepsilon(t)$ couples the modes, and depends on ε.

Sandwich (*) with $|L_\alpha\rangle$

$$\frac{d\ell_\alpha(t)}{dt} = -\langle X(t)|A L_\alpha\rangle + z(t)\,\ell_\alpha(t) + \varepsilon\, h_\alpha^L(t)$$

$$\mathbb{1} = \sum_{\beta=1}^{N} |R_\beta\rangle\langle L_\beta|$$

$$\frac{d\ell_\alpha(t)}{dt} = -\sum_\beta \lambda_\beta\, r_\beta(t)\langle L_\beta|L_\alpha\rangle + z(t)\,\ell_\alpha(t) + \varepsilon\, h_\alpha^L(t)$$

Again, inhomogeneous differential equation:

$$\frac{d\ell_\alpha(t)}{dt} = z(t)\,\ell_\alpha(t) + F_\alpha(t)$$

Where $F_\alpha(t) = -\sum_{\beta=1}^{N} \lambda_\beta\, r_\beta(t)\langle L_\beta|L_\alpha\rangle + \varepsilon\, h_\alpha^L(t)$.

Therefore:

$$\ell_\alpha(t) = \frac{\ell_\alpha(0)}{P_t^z(t)} + \int_0^t ds\, \frac{P_t^z(s)}{P_t^z(t)}\, F_\alpha(s)$$

$$= \frac{\ell_\alpha(0)}{P_t^z(t)} - \int_0^t ds\, \frac{P_t^z(s)}{P_t^z(t)} \sum_{\beta=1}^{N} \lambda_\beta\, r_\beta(s)\langle L_\beta|L_\alpha\rangle + \varepsilon \int_0^t ds\, \frac{P_t^z(s)}{P_t^z(t)}\, h_\alpha^L(s)$$

Ex. 2

Show that:

$$\rho_\alpha(t) = \sum_{\beta=1}^{N} \frac{r_\beta(0)\langle L_\beta | L_\alpha\rangle e^{-\lambda_\beta t}}{\Gamma_\varepsilon^2(t)} + \varepsilon \frac{1}{\Gamma_\varepsilon^2(t)}\left(D^R(t,h) + F^L(t,h)\right)$$

where:

$$D^R(t,h) = \sum_{\beta=1}^{N} \lambda_\beta \langle L_\beta | L_\alpha\rangle \int_0^t ds\, e^{-\lambda_\alpha s} \int_0^s du\, e^{\lambda_\alpha u}\, \Gamma_\varepsilon^2(u)\, h_\alpha^R(u)$$

$$F^L(t,h) = \int_0^t ds\, \Gamma_\varepsilon^2(s)\, h_\alpha^L(s)$$

Hints: Plug the expression for $r_\alpha(t)$ into $\rho_\alpha(t)$,
and use that $\rho_\alpha(0) = \sum_{\beta=1}^{N} r_\beta(0)\langle L_\beta | L_\alpha\rangle$

---

explicit dependence on overlaps!

When $\varepsilon \to 0$, $\rho_\alpha(t) = \sum_{\beta=1}^{N} \frac{r_\beta(0)\langle L_\beta | L_\alpha\rangle e^{-\lambda_\beta t}}{\Gamma_0^2(t)}$.

Correlation function and Lagrange multiplier

We focus on correlation function. We have

$$r_\alpha(t) = \frac{r_\alpha(0) e^{-\lambda_\alpha t}}{\Gamma_\varepsilon(t)} + \frac{\varepsilon}{\Gamma_\varepsilon(t)} \int_0^t ds\, e^{-\lambda_\alpha t + \lambda_\alpha s} \Gamma(s) h_\alpha^R(s)$$

$$\ell_\alpha(t') = \frac{1}{\Gamma_\varepsilon(t')} \sum_\beta r_\beta(0) e^{-\lambda_\beta t'} \langle L_\beta | L_\alpha \rangle + \frac{\varepsilon}{\Gamma_\varepsilon(t')} \left(D^R(t', h) + F^L(t', h) \right)$$

$$C_N(t,t') = \frac{1}{N} \sum_{\alpha,\beta} \frac{r_\alpha(0) e^{-\lambda_\alpha t}}{\Gamma_\varepsilon(t)} \frac{r_\beta(0)}{\Gamma_\varepsilon(t')} \langle L_\beta | L_\alpha \rangle e^{-\lambda_\beta t'} + \frac{\varepsilon\, a_h(t,t')}{\Gamma_\varepsilon(t)\Gamma_\varepsilon(t')}$$

where

$$a_h(t,t') = \sum_{\beta=1}^N r_\beta(0) e^{-\lambda_\beta t'} \langle L_\beta | L_\alpha \rangle \int_0^t ds\, e^{-\lambda_\alpha t + \lambda_\alpha s} \Gamma_\varepsilon(s) h_\alpha^R(s)$$

$$+ r_\alpha(0) e^{-\lambda_\alpha t} \left(D^R(t') + F^L(t') \right) + \mathcal{O}(\varepsilon^2)$$

The spherical constraint is $C(t,t) = 1$. For $\varepsilon \to 0$:

$$\Longrightarrow \Gamma_0(t) = \left(\frac{1}{N} \sum_{\beta,\alpha} r_\alpha(0) r_\beta(0) e^{-(\lambda_\alpha + \lambda_\beta)t} \langle L_\beta | L_\alpha \rangle \right)^{1/2}$$

Thus: $$C_N(t,t') = \frac{\frac{1}{N} \sum_{\alpha,\beta} r_\alpha(0) r_\beta(0) e^{-\lambda_\alpha t - \lambda_\beta t'} \langle L_\beta | L_\alpha \rangle}{\Gamma_0(t) \Gamma_0(t')}$$

Comments:

(i) One gets $z(t)$ from $z(t) = -\Gamma'(t)/\Gamma(t)$

(ii) When $\varepsilon > 0$, $\Gamma_\varepsilon(t)$ has a more complicated expression: needed to compute response!

$$\Gamma_\varepsilon(t) = \left(\Gamma_0^2(t) + \varepsilon\, a(t,t)\right)^{1/2}$$

The response function

$$r_\alpha(t) = \frac{r_\alpha(0)\, e^{-\lambda_\alpha t}}{\Gamma_\varepsilon(t)} + \varepsilon \int_0^t ds\, e^{-\lambda_\alpha (t-s)} \frac{\Gamma_\varepsilon(s)}{\Gamma_\varepsilon(t)} h_\alpha^R(s)$$

Use that $h_\alpha^R = \langle h | R_\alpha \rangle = \sum_{i=1}^N \langle h | i \rangle \langle i | R_\alpha \rangle$

Need: $$\left.\frac{\delta r_\alpha(t)}{\delta[\varepsilon h_i(t')]}\right|_{\varepsilon \to 0} = -\frac{r_\alpha(0)\, e^{-\lambda_\alpha t}}{\Gamma_0^2(t)} \frac{\delta \Gamma_\varepsilon(t)}{\delta[\varepsilon h_i(t')]} + \left.\int_0^t ds\, e^{-\lambda_\alpha (t-s)} \frac{\Gamma_0(s)}{\Gamma_0(t)} \langle i | R_\alpha \rangle\, \delta(s-t')\right|_{\varepsilon \to 0}$$

$$= -\frac{r_\alpha(0)\, e^{-\lambda_\alpha t}}{\Gamma_0^2(t)} \left.\frac{\delta \Gamma_\varepsilon(t)}{\delta[\varepsilon h_i(t')]}\right|_{\varepsilon \to 0} + \theta(t-t')\, e^{-\lambda_\alpha (t-t')} \frac{\Gamma_0(t')}{\Gamma_0(t)} \langle i | R_\alpha \rangle$$

EX. 3 Linear Response. To proceed, need to compute $\delta\Gamma$. Show that:

$$\left.\frac{\delta \Gamma_\varepsilon^2(t)}{\delta[\varepsilon h_i(t')]}\right|_{\varepsilon\to 0} = -\frac{\Theta(t-t')\,\Gamma_0^2(t')}{N\,\Gamma_0^2(t)} \sum_{\beta=1}^{N} r_\beta(0)\, e^{-\lambda_\beta(2t-t')} \langle L_\beta | i \rangle$$

Combining into: $R_N(t,t') = \frac{1}{N} \lim_{\varepsilon\to 0} \sum_{\alpha,i} \frac{\delta r_\alpha(t)}{\delta[\varepsilon h_i(t')]} \langle i | L_\alpha \rangle$

$$\Rightarrow R_N(t,t') = \frac{1}{N} \sum_{\alpha=1}^{N} \theta(t-t')\, e^{-\lambda_\alpha(t-t')} \frac{\Gamma_0^2(t')}{\Gamma_0^2(t)} +$$

$$+ \frac{1}{N} \sum_{\alpha,\beta} \frac{\theta(t-t')}{N} \frac{\Gamma_0^2(t')}{\Gamma_0^3(t)}\, r_\beta(0)\, r_\alpha(0)\, e^{-\lambda_\alpha t}\, e^{-\lambda_\beta(2t-t')}$$

II.B. THE LARGE-TIME LIMIT

Both the correlation and response depend on spectral properties of matrix A: eigenvalues, and eigenvectors overlaps $\langle L_\beta | L_\alpha \rangle$.

Depending on realization of A, different behaviours.

Some general facts? Take $A = A^T$.

$\Longrightarrow$ Focus: the symmetric case ($\alpha = 1$)

In this case, $\langle L_\beta | L_\alpha \rangle = \delta_{\alpha\beta}$ and the dependence on overlaps disappears:

$$C_N(t,t') = \frac{1}{N} \sum_{\alpha=1}^{N} r_\alpha^2(0)\, e^{-\lambda_\alpha (t+t')} \Big/ \Gamma_0^2(t)\, \Gamma_0^2(t')$$

$$R_N(t,t') = \frac{1}{N}\, \theta(t-t') \frac{\Gamma_0^2(t')}{\Gamma_0^2(t)} \sum_{\alpha=1}^{N} e^{-\lambda_\alpha (t-t')} \left(1 + \frac{1}{N} \frac{\Gamma_0(t')}{\Gamma_0^2(t)} r_\alpha^2(0)\, e^{-2\lambda_\alpha t'} \right)$$

Take $N=2$.

Assume evalues (they are real!) satisfy $\lambda_1 < \lambda_2$.

"SPECTRAL GAP": $\Delta = \lambda_2 - \lambda_1$.

Take $t' \to t_w$, $t \to t_w + \tau$. Then when $t_w \gg 1$:

$$C(t_w+\tau, t_w) = 1 - \frac{A_0}{2}\left(1 - e^{-\Delta\tau}\right)^2 e^{-2\Delta t_w} + \dots$$

$$R(t_w+\tau, t_w) = \frac{\theta(\tau)\, e^{-\Delta\tau}}{2} + \dots$$

where $\Delta = \lambda_2 - \lambda_1$, $A_0 = r_2^2(0)/r_1^2(0)$

How to see this? E.g. for correlation:

$$C(t,t') = \frac{\frac{1}{2}\sum_{\alpha=1}^{2} r_\alpha^2(0)\, e^{-\lambda_\alpha(t+t')}}{P(t)\,P(t')} =$$

$$= \frac{1 + A_0 e^{-\Delta(t+t')}}{\left(1+A_0 e^{-2\Delta t}\right)^{1/2}\left(1+A_0 e^{-2\Delta t'}\right)^{1/2}} \qquad \Delta = \lambda_2 - \lambda_1$$

What this tells us:

1. The system relaxes to a configuration at $t_w \to \infty$, fixed point of dynamics: $C(t_w+\tau, t_w) \xrightarrow{t_w \to \infty} 1$.

2. Correlation & Response converge exponentially, $\tau_{rel} = 1/\Delta$ where $\Delta = \lambda_2 - \lambda_1$ spectral gap

3. The $t_w \to \infty$ fixed point is predictable "from statics"

Dynamics: $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$ with $\vec{F}(\vec{x}) = -A\vec{x} + z\vec{x}$.

e value equation

Fixed points $\vec{x}^*$ satisfy $\vec{F}(\vec{x}^*) = 0 \Rightarrow A\vec{x}^* = z\vec{x}^*$

$\Longrightarrow \vec{x}^*_{\alpha,\pm} = \pm\sqrt{2}\,\vec{R}_\alpha$ are fixed points with $z^* = \lambda_\alpha$.

There are 4: which one attracts the dynamics at $t_w \gg 1$?

Recall: A is symmetric!

$$\Longrightarrow F_i(\vec{x}) = -\sum_j A^T_{ij} x_j + z x_i = \overbrace{-\frac{\partial}{\partial x_i}\mathcal{E}_z(\vec{x})}^{\text{"gradient descent"}}$$ DISSIPATIVE DYNAMICS

Energy function on sphere: $\mathcal{E}_z(\vec{x}) = \frac{1}{2}\vec{x}^T A \vec{x} - \frac{1}{2} z \left(\|\vec{x}\|^2 - 2\right)$

- Fixed points of the dynamics are stationary points of this function (maxima, minima, saddles).

- For $N=2$, two are minima and two are maxima:

$$\underbrace{\mathcal{H}_{ij}(\vec{x}^*_\beta)}_{\text{Hessian at Stat. point}} = \frac{\partial^2 \mathcal{E}(\vec{x}^*_\beta)}{\partial x_i \partial x_j} = A_{ij} - \lambda_\beta$$

$\beta = 1$: Hessian has evalues 0, and $\Delta = \lambda_2 - \lambda_1 > 0$
$\Rightarrow \vec{x}^*_{1,\pm}$ are minima

$\beta = 2$: Hessian has evalues 0, and $-\Delta = \lambda_1 - \lambda_2 < 0$
$\Rightarrow \vec{x}^*_{2,\pm}$ are maxima

- Moreover, $z(t) = -\frac{\vec{f}(\vec{x}) \cdot \vec{x}}{2} = \frac{1}{2} \vec{x}^T A \vec{x} = \mathcal{E}(\vec{x})$ ISTANTANEOUS ENERGY

$\mathcal{E}(\vec{x}_{\beta,\pm}) = \lambda_\beta$

$$z(t) = \frac{1}{2} \frac{\sum_\alpha \lambda_\alpha x_\alpha^2(0) e^{-2\lambda_\alpha t}}{\frac{1}{2} \sum_\beta x_\beta^2(0) e^{-2\lambda_\beta t}} = \frac{\lambda_1 + \lambda_2 A_0 e^{-2\Delta t}}{1 + A_0 e^{-2\Delta t}}$$

$$\xrightarrow{t \gg 1} \lambda_1 + A_0 \Delta e^{-2\Delta t}$$

At $t_w \to \infty$, the system is converging to the Ground-State: eigenvector associated to minimal eigenvalue.

Symmetry: two ground states, initial conditions select which one

Indeed: $z(t) = \mathcal{E}(\vec{x}(t)) \xrightarrow{t\to\infty} \lambda_1$

$$\vec{x}(t) \xrightarrow{t\to\infty} \sqrt{2}\ \mathrm{sign}(r_1(0))\, \vec{R}_1$$

4. One can say that system "equilibrates" at $T=0$: reaches the Ground State. However, This is not "equilibrium-like" dynamics at finite t_w. The correlation is not T.T.I, no FDT...
[no noise to compensate dissipation]

Ex. 4 Check that if white noise is added to the dynamics,

$$\frac{dx_i(t)}{dt} = -(A\vec{x})_i + z(t)x_i + \varepsilon h_i(t) + \sqrt{2T}\,\eta_i(t)$$

The two level system equilibrates with bath, and:

$$C(t_w+\tau, t_w) \overset{t_w \gg 1}{\sim} C_{eq}(\tau) \,, \quad R(t_w+\tau, t_w) \overset{t_w \gg 1}{\sim} R_{eq}(\tau)$$

with the FDT relation: $R_{eq}(\tau) = -\beta\, \theta(\tau) \frac{\partial C_{eq}(\tau)}{\partial \tau}$

Show that the decay of $C_{eq}(\tau)$ is controlled by both (ΔT)

Hint: expand on eigenbasis again.

Notice: each mode obeys Ornstein-Uhlenbeck equation at large time: $\frac{dx_\alpha}{dt} = (-\lambda_\alpha + z_\infty) x_\alpha + \eta_\alpha$.

Limits $T \to 0$ and $t \to \infty$ do not commute.

In summary, for N finite:

Q1. The dynamics is non-equilibrium relaxation to a fixed point. For $\alpha = 1$, convergence contolled by spectral gap Δ.

Q2. With non-reciprocity, need info on evector overlaps.

Q3. $\alpha = 1$: the fixed point is the Ground State of associated energy landscape $\mathcal{E}(\vec{x})$.

? How this changes when J_{ij} are random, and $N \gg 1$? In principle, all fluctuates & expect no sharp answer, need to deal with distributions... but:

- When $N \to \infty$, certain dynamical quantities converge to a deterministic limit: CONCENTRATION!

$$\lim_{N \to \infty} \underbrace{C_N(t,t')}_{\text{distributed}} = \underbrace{C_\infty(t,t')}_{\text{deterministic.}}$$

- Behaviour may depend only on global properties of distribution of J_{ij}: UNIVERSALITY!

III. LINEAR FORCES: LARGE-N

Goals of Part III:

1. Large N properties: averaging out (initial conditions) and concentration (disorder)
2. Spectral properties of random matrices: RMT
3. Reciprocal interactions: separation of timescales, aging, universality, landscape interpretation
4. Non-reciprocal interactions: dynamics speed-up

High-D system now: $N \to \infty$.

$$\frac{dx_i(t)}{dt} = \frac{\sigma}{\sqrt{N}} \sum_{j=1}^{N} J_{ij} x_j(t) + \frac{\mu}{N}\left(\sum_j x_j\right) + z(t)\, x_i(t) + \varepsilon\, \eta_i(t) \quad (*)$$

As before, $A = -\sigma/\sqrt{N}\, J^T - \frac{\mu}{N} \vec{1}\,\vec{1}^T$, and

$$z(t) = \frac{1}{N} \vec{x}^T(t)\, A\, \vec{x}(t) \quad (\varepsilon \to 0)$$

Noiseless dynamics (below): STARIOLO, METZ, 2026

With noise: CRISANTI, SOMPOLINSKY 1987

III. A. "TYPICAL" INITIAL CONDITIONS

$$C_N(t,t') = \frac{1}{N}\sum_{i=1}^{N} x_i(t)x_i(t') = \frac{1}{N}\sum_{\alpha,\beta=1}^{N} \frac{r_\alpha(0) r_\beta(0) e^{-\lambda_\alpha t} e^{-\lambda_\beta t'}}{\Gamma_0(t)\Gamma_0(t')} \langle L_\beta | L_\alpha \rangle$$

with $\Gamma_0^2(t) = \left(\frac{1}{N} \sum_{\alpha,\beta} r_\alpha(0) r_\beta(0) e^{-(\lambda_\alpha+\lambda_\beta)t} \langle L_\beta | L_\alpha \rangle \right)$

and similarly:

$$R_N(t,t') = \frac{1}{N}\Theta(t-t')\frac{\Gamma_0(t')}{\Gamma_0(t)}\sum_{\alpha=1}^{N} e^{-\lambda_\alpha(t-t')}\left[1+\frac{\Gamma_0(t')}{\Gamma_0^2(t)} S_N(t,t')\right]$$

$$S_N(t,t') = \frac{1}{N}\sum_{\beta=1}^{N} r_\beta(0) r_\alpha(0) e^{-\lambda_\alpha t' - \lambda_\beta(2t-t')} \langle L_\beta | L_\alpha \rangle$$

When $N=2$, memory of initial conditions at any t_w.

INITIAL CONDITIONS: $N \gg 1$ AND TYPICALITY

Consider A fixed: fixed quenched randomness.

We consider dynamics from "random initial conditions": assume $\vec{x}(0)$ extracted randomly (uniform measure) on the sphere $\mathcal{C}_N$. With respect to this distribution,

$$\overline{\otimes} := \frac{1}{|\mathcal{C}_N|}\int_{\mathcal{C}_N} d\vec{x}(0)\ \otimes$$

Quantities like $r_\alpha(0)\, r_\beta(0) = \langle R_\alpha | x(0)\rangle\langle x(0)|R_\beta\rangle$ depend on $\vec{x}(0)$, and fluctuate. However, they are large sums of independent, fluctuating terms:

$$\langle R_\alpha | x(0)\rangle\langle x(0)|R_\beta\rangle = \sum_{i,j=1}^{N} R_\alpha^i\, x(0)_i\, R_\beta^j\, x(0)_j$$

independent: R_α are given vectors, $x(0)$ isotropic

When $N \gg 1$, fluctuations over $\vec{x}(0)$ are suppressed

(Law Large Numbers) and

$$\lim_{N\to\infty} \left(r_\alpha(0)\, r_\beta(0)\right) = \lim_{N\to\infty} \sum_{i,j}^{N} R_\alpha^i x_0^i R_\beta^j x_0^j = \lim_{N\to\infty} \sum_{i,j=1}^{N} R_\alpha^i R_\beta^j\ \overline{x_0^i\, x_0^j}$$

Now, when $N \gg 1$ random vectors on sphere are statistically equivalent to random Gaussian vectors:

$$\overline{x_0^i} = 0 \quad \text{and} \quad \overline{x_0^i\, x_0^j} = \delta_{ij}$$

(normalization condition not relevant when $N\to\infty$: norm concentrates to 1 by law large numbers).

and thus $r_\alpha(0)\, r_\beta(0) \xrightarrow{N\to\infty} \langle R_\alpha | R_\beta\rangle$.

Therefore:

$$\lim_{N\to\infty} \mathcal{C}_N(t,t') = \lim_{N\to\infty} \frac{\frac{1}{N}\sum_{\alpha,\beta} e^{-\lambda_\alpha t - \lambda_\beta t'} \langle L_\beta | L_\alpha \rangle \; \overline{r_\alpha(0) r_\beta(0)}}{\overline{\Gamma_0(t)} \cdot \overline{\Gamma_0(t')}}$$

Define the overlaps: $\quad O_{\alpha\beta} = \langle L_\beta | L_\alpha \rangle \langle R_\beta | R_\alpha \rangle$

$$\lim_{N\to\infty} C_N(t,t') = \lim_{N\to\infty} \frac{\frac{1}{N}\sum_{\alpha,\beta} e^{-\lambda_\alpha t} e^{-\lambda_\beta t'} O_{\alpha\beta}}{\left(\frac{1}{N}\sum_{\alpha,\beta} e^{-(\lambda_\alpha+\lambda_\beta)t} O_{\alpha\beta}\right)^{1/2} \left(\frac{1}{N}\sum_{\alpha,\beta} e^{-(\lambda_\alpha+\lambda_\beta)t'} O_{\alpha\beta}\right)}$$

Define $W_N(t,t') := \frac{1}{N}\sum_{\alpha,\beta} e^{-\lambda_\alpha t - \lambda_\beta t'} O_{\alpha\beta}$

and $D_N(z_1,z_2) = \frac{1}{N}\sum_{\alpha,\beta} O_{\alpha\beta}\, \delta^2(z_1-\lambda_\alpha)\, \delta^2(z_2-\lambda_\beta)$

Then: $C_N(t,t') = \dfrac{W_N(t,t')}{\sqrt{W_N(t,t)\, W_N(t',t')}}$

with $W_N(t,t') = \int dz_1\, dz_2\, e^{-z_1 t - z_2 t'} D_N(z_1,z_2)$

After averaging over initial conditions: everything is a function of the evalues & evectors of matrix J. Can we say something about them, when $N \to \infty$?

$\Rightarrow$ RANDOM MATRIX THEORY.

III.B. RANDOM MATRICES: SOME FACTS

Matrix A from Elliptic Ensemble: Gaussian,

$$\mathbb{E}[A_{lm}] = -\mu/N$$

$$\mathbb{E}[A_{lm} A_{kn}] - (\mu/N)^2 = \frac{\sigma^2}{N}\left(\delta_{lk}\delta_{mn} + \alpha\, \delta_{ln}\delta_{mk}\right)$$

Parameters: σ, α, μ.

And $\mathbb{E}[\cdot]$ is average over the ensemble $\left(\mathbb{P}_{N,\alpha}(A)\right)$

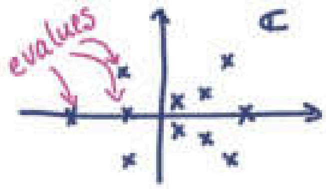
Eigensystem: $\{\lambda_\alpha \in \mathbb{C}, (\vec{R}_\alpha, \vec{L}_\alpha)\}_{\alpha=1}^{N}$.

Statistical properties of these objects?

Eigenvalue Distribution

N finite: $\nu_N(z) = \frac{1}{N} \sum_{\alpha=1}^{N} \delta(z - \lambda_\alpha)$

empirical distribution



When $N \to \infty$, $d\nu_N(z) \overset{N \gg 1}{\sim} \rho_N(z)\,dz + \frac{1}{N} \sum_{i=1,2,\dots} \delta(z - \lambda_{i,N}^{(iso)})\,dz$

$\rho_N(z)$ describes the distribution where eigenvalues accumulate
$\lambda_i^{(iso)}$ denotes eventual isolated eigenvalues, "outliers"

(a) When $N \to \infty$, $\rho_N(z)$ is self-averaging

$$\lim_{N\to\infty} \underbrace{\rho_N(z)}_{\text{random}} = \underbrace{\rho_\infty(z)}_{\text{deterministic}} = \lim_{N\to\infty} \mathbb{E}[\rho_N(z)]$$

Means: $\frac{1}{N} \sum_{\alpha=1}^{N} h(\lambda_\alpha) \xrightarrow{N\to\infty} \int dz\, \rho_\infty(z)\, h(z)$

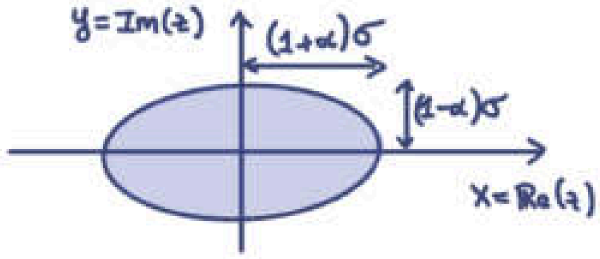
(b) For Elliptic matrices, "ELLIPTIC LAW"

$$\rho_\infty(z) = \rho_\alpha(x,y) = \frac{1}{\pi \sigma^2 (1-\alpha^2)} \Theta\left[1 - \frac{x^2}{\sigma^2(1+\alpha)^2} - \frac{y^2}{\sigma^2(1-\alpha)^2}\right]$$

uniform density

compact support

Notice: no dependence on μ!



SOMMERS, CRISANTI, SOMPOLINSY & STEIN 1988

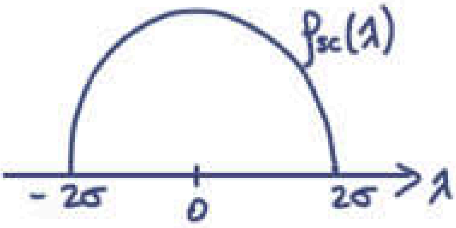
GIRKO 1986

Two special cases:

$\underline{\alpha = 0}$: support is a circle $\Rightarrow$ "circular law"
Ginibre ensemble GINIBRE 1965, GIRKO 1984

$\underline{\alpha = 1}$: support is the real axis, and density is "Wigner semicircular law" $\Rightarrow \rho_{sc}(\lambda)$

$$\rho_\alpha(x,y) \xrightarrow{\alpha \to 1} \rho_{sc} = \frac{1}{2\pi}\sqrt{4\sigma^2 - x^2}\,\Theta\left(2\sigma - |x|\right)\delta(y)$$



The matrix ensemble when $\alpha \to 1$ is called GOE: Gaussian Orthogonal Ensemble

WIGNER 1958

Wigner law. In distribution:

$$\int dx\, dy\, \rho_\alpha(x,y)\, \phi(x,y) = \int_{-(1+\alpha)\sigma}^{(1+\alpha)\sigma} dx \int_{|y|<\sigma(1-\alpha)\sqrt{1-\frac{x^2}{1+\alpha}}} dy\, \rho_\alpha(x,y)\, \phi(x,y)$$

when $\alpha \to 1$, $|y| \leq \varepsilon$ and $\phi(x,y) \sim \phi(x,\varepsilon) \to \phi(x,0)$

Moreover, the marginal:

$$\int dy\, \rho_\alpha(x,y) = \frac{2}{\pi\sigma(1+\alpha)}\sqrt{1-\frac{x^2}{\sigma^2(1+\alpha)^2}} \xrightarrow{\alpha\to 1} \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}$$

(c) If $\mu \neq 0$, an isolated eigenvalue can exist when $|\mu| > \sigma$.

When $N \to \infty$, it converges to the typical value:

$$\lim_{N\to\infty} \lambda_N^{(iso)} = \lambda_\infty^{(iso)} = -\left(\mu + \frac{\alpha\sigma^2}{\mu}\right)$$

[At $\mu = \sigma$, right at edge: $\lambda_\infty^{(iso)} = -\sigma(1+\alpha)$]

O' ROURKE, RENFREW 2014

For $\alpha = 1$, analysis of fluctuations of $\lambda_N^{(iso)}$ made by

BAIK, BEN AROUS, PÉCHÉ 2005

Now these transitions called "BBP transitions".

EIGENVECTORS STATISTICS

Two point correlator at N finite:

$$D_N(z_1,z_2) = \frac{1}{N}\sum_{\alpha,\beta=1}^{N} O_{\alpha\beta}\, \delta(z_1-\lambda_\alpha)\,\delta(z_2-\lambda_\beta)$$

Correlation between eigenvectors at two points z_1, z_2.

When $N\to\infty$, this is also self-averaging.

Complex Ginibre ensemble

For this ensemble, explicit result:

$$\lim_{N\to\infty} D_N(z_1,z_2) = D(z_1,z_2) = \frac{(1+\alpha^2) z_1 z_2^* - \sigma^2(1-\alpha^2)^2 - \alpha\left[z_1^2 + (z_2^*)^2\right]}{\pi^2\sigma^2(1-\alpha^2)\,|z_1-z_2|^4}$$

for z_1, z_2 in ellipse

MEHLIG, CHALKER 2000 CHALKER, MEHLIG 1998
BOURGADE, DUBACH 2020

:

Real Ginibre ensemble

The behavior seems same for eigenvalues away from real axis. Shown for diagonal ($\alpha = \beta$)

WÜRFEL, CRUMPTON, FYODOROV 2023

WURFEL, CRUMTON, FYODOROV 2024, 2025

Notice: $\lim_{N\to\infty} W_N(t,t') = \int_{\text{ellipse}} dz_1\, dz_2\, D(z_1,z_2)\, e^{-z_1 t}\, e^{-z_2 t'}$

huge weight at $z_1 \simeq z_2$

At large time should favor eigenvalues with large $|\mathrm{Re}\, z|$

Contibution also from bulk evalues when $t, t' \gg 1$.

$\Rightarrow$ consequences for dynamics!

III.C. BACK TO DYNAMICS: RECIPROCAL

Set $\sigma^2 = 1$. Then $A = -J^T = -J$ is a GOE matrix.

$$W_N(t,t') = \frac{1}{N}\sum_{\alpha,\beta} e^{-\lambda_\alpha t - \lambda_\beta t'} O_{\alpha\beta} = \frac{1}{N}\sum_{\alpha=1}^{N} e^{-\lambda_\alpha(t+t')}$$

$$= \int d\nu_N(\lambda)\, e^{-\lambda(t+t')}$$

empirical density of evalues

When $N\to\infty$, this is self-averaging and

$$\lim_{N\to\infty} W_N(t,t') = \int \rho_{sc}(\lambda)\, d\lambda\, e^{-\lambda(t+t')} = \frac{I_1(2(t+t'))}{t+t'}$$

$I_1(u)$ = modified Bessel function of first Kind

$$C_N(t_w+\tau, t_w) \xrightarrow{N\to\infty} \frac{I_1(2(2t_w+\tau))}{2t_w+\tau}\sqrt{\frac{4t_w(t_w+\tau)}{I_1(4t_w)\, I_1(4(t_w+\tau))}}$$

this describes large-N dynamics at all times.

MEAN-FIELD DYNAMICS

Asymptotics: $t_w \gg 1$?

We are interested in $t_w \gg 1$ (asymptotics). In this limit, if dynamics is not T.T.I. (dependence on t_w not disappears): dynamics again out-of-equilibrium.

Regime 1 [R1]: τ small, $\tau/t_w \ll 1$

Regime 2 [R2]: τ large, $\tau/t_w \gg 1$

In [R1]: Short time differences

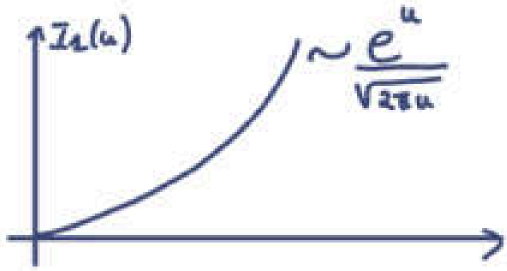
For τ small, $C(t_w+\tau, t_w) \simeq 1$. At short time difference, do not decorrelate. memory..

$$\lim_{\tau/t_w \to 0} C(t_w+\tau, t_w) = 1$$

Expanding: $C(t_w+\tau, t_w) = 1 - \frac{1}{4}\, \tau^2/t_w + \dots$

decorrelate at $\tau \sim \sqrt{t_w}$

In [R2]: large time-differences



$$C_N(t_w+\tau, t_w) \xrightarrow{N\to\infty} \frac{I_1(2(2t_w+\tau))}{2t_w+\tau}\sqrt{\frac{4t_w(t_w+\tau)}{I_1(4t_w)\, I_1(4(t_w+\tau))}}$$

Asymptotics:

$$\frac{e^{2(2t_w+\tau)}}{\sqrt{2(2t_w+\tau)}}\frac{1}{\sqrt{2\pi}}\frac{1}{2t_w+\tau}\frac{2\sqrt{t_w(t_w+\tau)}}{e^{2t_w+2t_w+2\tau}}\left(2\pi\cdot 4\sqrt{t_w(t_w+\tau)}\right)^{1/2}$$

The exponentials cancel and:

$$2\sqrt{2}\sqrt{\frac{t_w(t_w+\tau)}{2t_w+\tau}}\frac{1}{2t_w+\tau}\left(t_w(t_w+\tau)\right)^{1/4}=$$

$$=2\sqrt{2}\frac{1}{(2t_w+\tau)^{3/2}}\left(t_w(t_w+\tau)\right)^{3/4}=\frac{2\sqrt{2}\,\tau^{3/4}}{\tau^{3/2}}\frac{t_w^{3/4}\,(1+t_w/\tau)^{3/4}}{(1+2t_w/\tau)^{3/2}}$$

$$=2\sqrt{2}\left(\frac{t_w}{\tau}\right)^{3/4}+\dots$$ slow, algebraic decay

$$\lim_{\tau\to\infty} C(t_w+\tau, t_w)=0$$

This discussed in: CUGLIANDOLO, DEAN 1995

let us comment these findings.

A SLOW DYNAMICS.

(1) At any finite t_w, the system is not at fixed point! Separation of timescales:

For all times $\tau \lesssim \sqrt{t_w}$ it "seems" at fixed point.

but as τ increases C decays to zero: system keeps moving and exploring new configurations...

(2) Never T.T.I: decay depends on (t_w/τ): this is called AGING: system slower as t_w grows. take more & more time to decorrelate...

(3) Slow, algebraic decay. Unlike $N=2$!

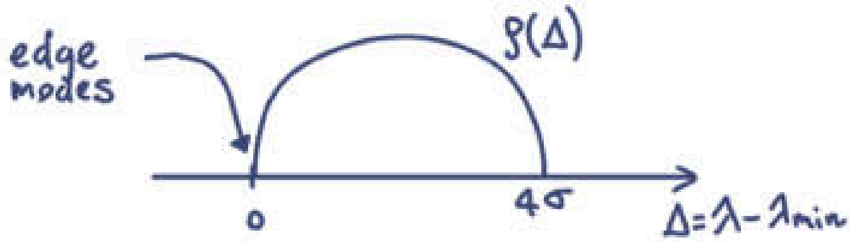
THE LARGE-N EFFECT

When $N=2$, system relaxes to Ground State with speed controlled by $\Delta = \lambda_2 - \lambda_1$.

Role of $N \gg 1$: Algebraic decay.

$$C_N(t_w+\tau, t_w) = \frac{1 + \sum_{\alpha\neq 1} e^{-(\lambda_\alpha - \lambda_1)(2t_w+\tau)}}{\left(1+\sum_{\alpha\neq 1} e^{-(\lambda_\alpha-\lambda_1)t_w}\right)^{1/2}\left(1+\sum_{\alpha\neq 1} e^{-(\lambda_\alpha-\lambda_1)(t_w+\tau)}\right)^{1/2}}$$

Modes with $(\lambda_\alpha - \lambda_1) \sim \mathcal{O}(1)$ becomes irrelevant when $t_w \gg 1$: modes with $(\lambda_\alpha - \lambda_1) \ll 1$ matter. These are "edge modes" when $N \to \infty$



The gaps $\Delta_\alpha = \lambda_\alpha - \lambda_1 \xrightarrow{N\to\infty} 0$: spectrum gapless!

In fact, $\begin{cases} \Delta_\alpha \sim 1/N & \text{if } \alpha \text{ in "bulk"} \\ \Delta_\alpha \sim 1/N^{2/3} & \text{if } \alpha \text{ in "edge"} \end{cases}$

$\Rightarrow$ No exponential decay, but slow power law.

THE RANDOMNESS: UNIVERSALITY

At large times, only eigenvalues at the edge contribute. The power 3/4 depends on how $\rho(\lambda)$ vanishes at $\lambda_1 = \lambda_{min}$.

Can show that if $\rho_\infty(\lambda) \overset{\lambda = \lambda_{min}}{\sim} (\lambda - \lambda_{min})^\gamma$

then $C(t_w + \tau, t_w) \sim \left(t_w/\tau\right)^{\frac{\gamma+1}{2}}$

The power does not depend on details of evalue distribution (that depend on $P(J)$)

THE LANDSCAPE INTERPRETATION

Dynamics is Gradient Descent on Landscape.

$$\frac{dx_i(t)}{dt} = -\frac{\partial \mathcal{E}_z(\vec{x})}{\partial x_i} \quad \begin{cases} \mathcal{E}_z(\vec{x}) = \frac{1}{2}\vec{x}^T \cdot A\,\vec{x} - \frac{z}{2}\left(\|\vec{x}\|^2 - N\right) \\ z(\vec{x}) = \frac{1}{N}\vec{x}^T \cdot A \cdot \vec{x} = \frac{2\mathcal{E}(\vec{x})}{N} = 2\epsilon(\vec{x}) \end{cases}$$

energy density

Fixed points $\vec{x}^*$ solve $-A\vec{x}^* + z\vec{x}^* = 0$.

If $\vec{v}_\alpha$ is eigenvector of A with eigenvalue λ_α,
$\vec{x}_\alpha = \pm \sqrt{N}\, \vec{v}_\alpha$ is a fixed point with energy density $\epsilon_\alpha = \frac{\lambda_\alpha}{2} \Rightarrow$ there are 2N of them!

(a) As for N=2, all evectors are fixed points of the dynamics, and stationary points (minima, maxima, saddles) of $\mathcal{E}(\vec{x})$ of energy density $\epsilon = \lambda_\alpha/2$.

(b) Optimization dynamics should converge to global minimum of $\mathcal{E}(\vec{x})$, the Ground State $\Rightarrow$ Boltzmann equilibration at T=0. Ground state is minimal evector of A. Role of other evectors?

(c) All other eigenvectors are saddles of energy landscape. The lower is they energy, the lower is their index $\Rightarrow$ more and more "trapping" for dynamics. This gives GEOMETRICAL INTERPRETATION of aging. As time proceeds, descend in landscape and find it harder & harder to escape...

BOUCHAUD, CUGLIANDOLO, KURCHAN, MEZARD 1997

III.D. DYNAMICS: NON-RECIPROCITY

Again, $\sigma^2 = 1$.

$$W_N(t,t') = \frac{1}{N} \sum_{\alpha,\beta} e^{-\lambda_\alpha t - \lambda_\beta t'} O_{\alpha\beta}$$

$$= \int d^2 z_1 \int d^2 z_2 \, D_N(z_1, z_2) \, e^{-z_1 t - z_2 t'}$$

When $N \to \infty$, this is also self-averaging!

$$\lim_{N\to\infty} W_N(t,t') = \int d^2 z_1 \, d^2 z_2 \, D(z_1, z_2) \, e^{-z_1 t - z_2 t'}$$

This double integral can be done exactly (it is a bit of work in complex analysis....)

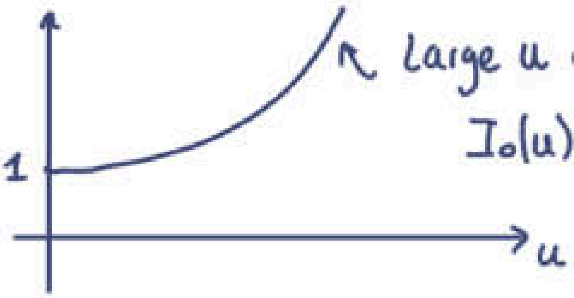
$\underline{\alpha = 0}$ $\quad W(t_w + \tau, t_w) = I_0\left(2\sqrt{t_w(t_w+\tau)}\right)$

product of two times, with a <u>square root</u>

$$W(t,t) = I_0(2t)$$

$I_0(u)$ = modified Bessel function of first Kind

Large- time asymptotics , $\tau / t_w \gg 1$.



large u asymptotics:

$$I_0(u) \sim \frac{e^u}{\sqrt{2\pi u}}$$

$$W(t_w+\tau, t_w) = \frac{e^{2\sqrt{t_w(t_w+\tau)}}}{\sqrt{4\pi t_w(t_w+\tau)}} \quad , \quad W(t_w,t_w) = \frac{e^{2t_w}}{\sqrt{4\pi t_w}}$$

$$C(t_w+\tau, t_w) \sim e^{2\sqrt{t_w(t_w+\tau)} - t_w - (t_w+\tau)} \sim e^{-\Delta\tau}$$

$$\Delta = 1 + \frac{2t_w}{\tau} - 2\sqrt{\frac{t_w}{\tau}\left(t_w/\tau + 1\right)}$$

exponential decay with varying gap!

$$\Delta\left(t_w/\tau\right) \xrightarrow{t_w/\tau \to 0} 1$$

$$\Delta(t_w/\tau) \overset{t_w/\tau \to \infty}{\sim} \frac{1}{4}\, 1/(t_w/\tau)$$

Notice: the gap comes from the eigenvectors' contribution: not only extremal eigenvalues matter, but also correlated pairs of evectors with same evalue $\approx$ near-diagonal sector.
Dynamics is not controlled only by spectral gap.

Generic α

$$C(t_w+t, t_w) \sim \left[\frac{(1+\eta)^2}{\eta}\frac{t_w}{\tau}\right]^{5/4} e^{-\Delta_\alpha(t_w/\tau)\tau}$$

with $\Delta_\alpha(t_w/\tau) = (1-\sqrt{\alpha})^2\left(1 - \frac{(1+\alpha)}{\sqrt{\alpha}}\frac{t_w}{\tau}\right)$

For arbitrary $\alpha \neq 1$, dynamics is faster (non-algebraic)

All this is discussed in STARIOLO, METZ 2026

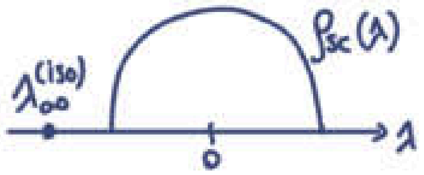
A TRANSITION

this is "mean-field dynamics"

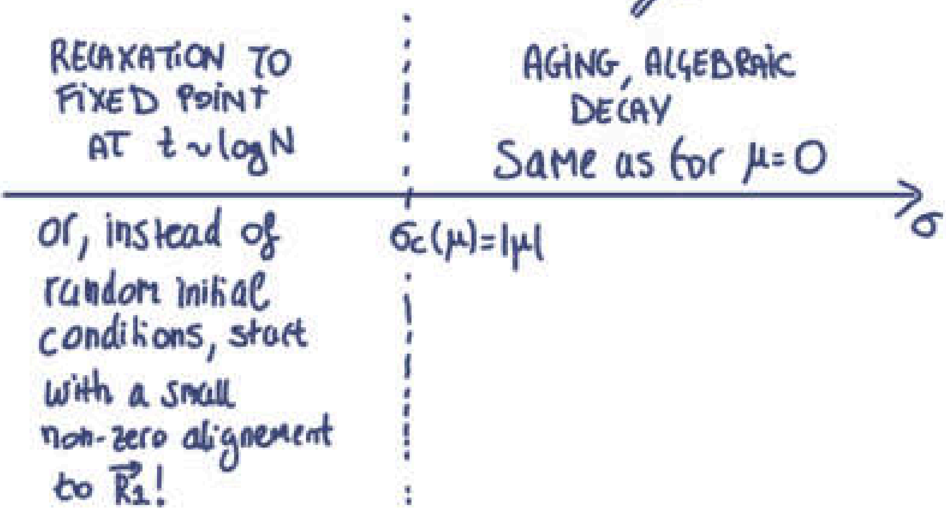
So far, no dependence on μ because took $N \to \infty$ before $t_w \to \infty$. When $\mu > 0$, a transition in the spectrum of A: at $|\mu| > \sigma$, an isolated evalue appears.

The above analysis of mean-field dynamics remains true (contribution of outlier negligible when $N \to \infty$).

Symmetric case ($\alpha = 1$): at times $t \sim \log N$, crossover when $|\mu| > \sigma$: system relaxes to ground state $\lambda_\infty^{(iso)}$, exponentially fast (system gapped)



Dynamics beyond mean-field ($\alpha = 1$): FYODOROV, PERRET, SCHEHR 2015



$\Longrightarrow$ Back to our Questions.

Q1. Out-of-equilibrium dynamics with separation of timescales, aging, and (i) algebraic ($\alpha = 1$) or (ii) exponential ($\alpha \neq 1$) decay.

Q2. Non-reciprocity does not simply speed-up relaxation: it changes the <u>kind</u> of non-equilibrium. Destroys algebraic decay in favor of exponential.
The point $\alpha = 1$ is singular.

$\rightarrow$ Also entropy production is faster:
FYODOROV, GUDOWSKA, NOWAK, TARNOWSKI '25

Q3. A role for fixed points?
Fixed points are left eigenvectors associated to real eigenvalues.
The number of real eigenvalues $\sim \sqrt{N}$: much less then $\alpha = 1$.
Is this why dynamics speeds up?

STARIOLO, METZ 2026

IV. LARGE N: NON-LINEAR FORCES

Goals of Part IV:

1. Dynamics at $N\to\infty$: DMFT self-consistent process.
2. Reciprocal interactions: aging, but rugged landscape.
3. Non-reciprocity: chaos, description of asymptotic regime, dynamical phase diagram.

Consider now the model with non-linear, random non-reciprocal forces:

$$f_i(\vec{x}) = \frac{\mu}{N}\sum_{j=1}^{N} x_j + \frac{\sigma}{N^{\frac{p}{2}}}\sum_{i_2,\dots,i_p} J_{i i_2 \dots i_p} x_{i_2}\cdots x_{i_p}$$

with $p=3$. Meaning:

$$\frac{dx_i(t)}{dt} = +\frac{\sigma}{N}\sum_{j,k=1}^{N} J_{ijk}\, x_j x_k + \frac{\mu}{N}\sum_{j=1}^{N} x_j + z(t)\, x_i + \varepsilon\, h_i(t)$$

$$\mathbb{E}[J_{ijk} J_{\ell mn}] = (1-\alpha)\,\delta_{i\ell}\left[\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}\right] + \alpha \sum_{\pi} \delta_{i\pi(\ell)}\delta_{j\pi(m)}\delta_{k\pi(n)}$$

The spherical constraint again implies:

$$Z(t) = -\frac{1}{N}\left[\frac{\sigma}{N}\sum_{ijk} J_{ijk}\, x_i x_j x_k + \frac{\mu}{2N}\left(\sum_{j=1}^{N} x_j\right)^2\right]$$

In this case, can not decompose into modes... how to solve for dynamics?

IV.A. DYNAMICAL MEAN-FIELD THEORY

When $N \to \infty$, one can study dynamics in mean-field: DMFT - Dynamical Mean-Field Theory.

Recall: mean-field

Fully connected Ising: $E = -\frac{J}{N}\sum_{ij} x_i x_j - h\sum_i x_i$ with $x_i = \pm 1$.

Can think of as $E = \sum_i h_i^{eff}(\vec{x})\, x_i$

Effective field: $h_i^{eff}(\vec{x}) = h + J/N \sum_{j=1}^{N} x_j$ effect of all other spins on x_i

When N large: $h_i^{eff} \approx h^{eff} = h + J m$, $m = \frac{1}{N}\sum_{j=1}^{N} x_j$

Then, single spin problem with

$$P_\beta(x) = \frac{e^{\beta(h+Jm)x}}{Z_\beta} \Rightarrow \langle x \rangle = \tanh\left(\beta[h+Jm]\right)$$

Self-consistency: $m = \tanh(\beta[h + \alpha m])$

Applying same ideas to dynamics, can show that for $N \to \infty$ dynamics described in terms of one effective degree of freedom $x(t)$ that follows equation:

$$\frac{dx(t)}{dt} = F[x(t)] + \eta(t) + \varepsilon h(t)$$

(from "random" initial conditions)

EFFECTIVE FORCE:

$$F = z(t)\, x(t) + \mu\, m(t) + 4\alpha \int_0^t ds\, \sigma^2 C(t,s)\, R(t,s)\, x(s)$$

A "memory" term:
retarded self-interactions

Encodes the fact that if x_i changes at t', this affects other x_j, that in turn affect x_i at $t > t'$.
This term vanishes for $\alpha = 0$ (averages out)

$C(t,t')$ and $R(t,t')$ are correlation and response functions when $N \to \infty$:

$$C(t,t') = \lim_{N\to\infty} \frac{1}{N} \vec{x}^T(t)\cdot\vec{x}(t')$$

$$R(t,t') = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \frac{\delta x_i(t)}{\delta[\varepsilon h_i(t')]}$$

$$m(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i(t)$$

$\Rightarrow$ at N finite, functions are random. As $N\to\infty$, self-averaging!

EFFECTIVE NOISE

Accounts for fluctuations of the x_j. It is Gaussian, with:

$$\langle \eta(t)\rangle = 0, \quad \langle \eta(t)\eta(t')\rangle = 2\sigma^2 C^2(t,t')$$

and $\langle\ldots\rangle$ = expectation value on process.

SELF-CONSISTENCY

The single-site process is representative of the whole dynamics. Thus:

$$m_N(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \xrightarrow{N\to\infty} m(t) = \langle x(t)\rangle$$

$$C_N(t,t') = \frac{1}{N}\sum_{i=1}^{N} x_i(t)\, x_i(t') \longrightarrow C(t,t') = \langle x(t)\, x(t')\rangle$$

$$R_N(t,t') = \frac{1}{N}\sum_{i=1}^{N} \frac{\delta x_i(t)}{\delta[\varepsilon h_i(t')]} \longrightarrow R(t,t') = \left.\frac{\delta\langle x(t)\rangle}{\delta[\varepsilon h(t')]}\right|_{\varepsilon\to 0} = \left\langle \frac{\delta x(t)}{\delta\eta(t')}\right\rangle$$

$\Longrightarrow$ Self-consistent [dynamical!] equations for $m(t)$, $C(t,t')$, $R(t,t')$. Integro-differential equations.

IV.B. THE RECIPROCAL CASE

For $\mu=0$, DMFT equations studied in
CUGLIANDOLO, KURCHAN 1993

Review: BOUCHAUD, CUGLIANDOLO, KURCHAN, MEZARD 1997

- "Explicit" asymptotic solution (aging ansatz)

- The scenario in part "similar" to linear case: separation of timescales, aging $C(t_w+\tau, t_w) = C\left(h(t_w)/h(t)\right)$

- One difference: do not reach the ground state energy density at $t\to\infty$, but a higher one:
$$\lim_{t\to\infty} \left\langle \frac{\mathcal{E}(x(t))}{N} \right\rangle = \epsilon_{th} > \epsilon_{GS}$$

▨ Landscape interpretation: not only two minima and several saddles ($\Rightarrow p=2$), but exponentially many minima and saddles... the dynamics gets stuck in local minima much higher than grond state

exponentially-many minima $\longleftrightarrow$ stable fixed points

Geometry of landscape studied in detail, eg. with Kac-Rice.

Lecture notes focusing on this: VR, SCIPOST PHYSICS LECTURE NOTES, 102

IV.C. THE NON-RECIPROCAL CASE

Model with $\mu=0$: CUGLIANDOLO, KURCHAN, LE DOUSSAL, PELITI 1997
BERTHIER, BARRAT, KURCHAN 2000

Here: FOURNIER, PACCO, ROS, URBANI 2026

$\Longrightarrow$ Focus: the case $\alpha=0$

▨ EQUATIONS. No memory term:

$$\frac{dx(t)}{dt} = z(t)\,x(t) + \mu\, m(t) + \eta(t) + \varepsilon\, h(t)$$

The DMFT equations:

(1) $$\frac{dm(t)}{dt} = \left(z(t) + \mu\right) m(t)$$

(3) $$\frac{\delta}{\delta \eta(t')}\left(\frac{dx(t)}{dt}\right) = z(t)\,\frac{\delta x(t)}{\delta \eta(t')} + \delta(t-t')$$

$$\frac{\partial R(t,t')}{t} = z(t)\,R(t,t') + \delta(t-t')$$

(2) $$\frac{\partial C(t,t')}{\partial t} = \left\langle \left[z(t)x(t) + \mu\, m(t) + \eta(t)\right] x(t') \right\rangle$$

$$= z(t)\,C(t,t') + \mu\, m(t)\,m(t') + \langle \eta(t)\, x(t') \rangle$$

Show that if $\eta(t)$ is Gaussian noise with $\langle \eta(t)\rangle = 0$, $\langle \eta(t)\,\eta(t')\rangle = \Phi_1\left(C(t,t')\right)$

Then: $\langle \eta(t)\,x(t')\rangle = \int_0^{t'} ds\, \Phi_1\left(C(t,s)\right) \left\langle \frac{\delta x(t')}{\delta \eta(s)} \right\rangle$

And that $\left\langle \frac{\delta x(t')}{\delta \eta(s)} \right\rangle = R(t',s)$

Hints: use that $P[\eta(t)] \sim \exp\left\{ -\frac{1}{2} \int dt\, dt'\, \eta(t)\, \Phi_1^{-1}(t,t')\, \eta(t') \right\}$

and that $\frac{\delta P[\eta(t)]}{\delta \eta(s)} = P[\eta(t)]\left(- \int dt'\, \Phi_1^{-1}(s,t')\, \eta(t')\right)$.

Hint: Use integration by parts to show the identity.

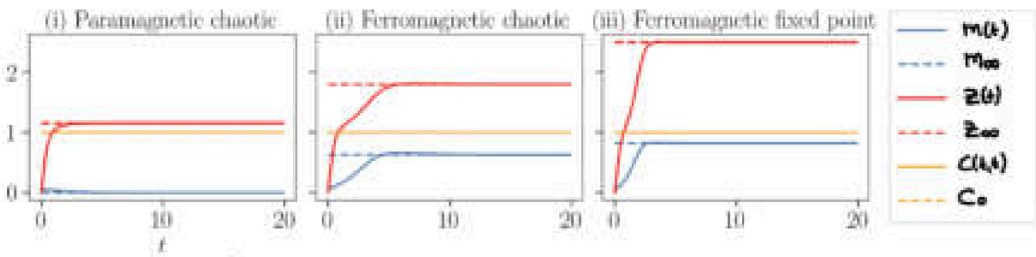
Then:

$$(3)\quad \frac{\partial C(t,t')}{\partial t} = z(t)C(t,t') + \mu(t)\, m(t) m(t') + 2\sigma^2 \int_0^{t'} ds\, C^2(t,s) R(t',s)$$

Usin that $C(t,t)=1$ $\left(\partial_t C(t,t)=0\right)$

$$(4)\quad z(t) = -2\sigma^2 \int_0^{t} ds\, [C(t,s)]^2 R(t,s) - \mu\, m^2(t)$$

▨ THE T.T.I. REGIME. Integrating equations numerically, one sees that after a time $t_0 \sim \mathcal{O}(1)$, dynamics becomes T.T.I:



$$C(t,t') \overset{t,t' \gg t_0}{\sim} C(\tau) = C(t-t')$$
$$R(t,t') \overset{t,t' \gg t_0}{\sim} R(\tau) = R(t-t')$$
$$z(t) \overset{t \gg t_0}{\sim} z_\infty\,, \quad m(t) \overset{t \gg t_0}{\sim} m_\infty$$

DMFT $\iff$ equations for $C(\tau), R(\tau), z_\infty, m_\infty$

Plug the T.T.I ansatz:

(1) $(z_\infty + \mu) m_\infty = 0$ $m_\infty = 0$ always a fixed point!

(3) $\partial_\tau R(\tau) = z_\infty R(\tau) + \delta(\tau) \Rightarrow R(\tau) = \theta(\tau) e^{z_\infty \tau}$

Causality: $R(\tau) = 0 \quad \tau < 0$

Jump condition from surce: $R(0^+) = 1$

(2) $\partial_\tau C(\tau) = z_\infty C(\tau) + \mu m_\infty^2 + 2\sigma^2 \int_0^{t'} ds\, C^2(t,s) R(t',s)$

change of variables: $u = t' - s$

$2\sigma^2 \int_0^{t'} du\, C^2(t, t'-u) R(t', t'-u)$

$\sim 2\sigma^2 \int_0^{\infty} du\, C^2(u+\tau) R(u)$

$\Rightarrow \partial_\tau C(\tau) = z_\infty C(\tau) + \mu m_\infty^2 + 2\sigma^2 \int_0^{\infty} du\, C^2(u+\tau) e^{z_\infty u}$

By spherical constraint: $C_0 = \lim_{\tau \to 0} C(\tau) = 1$

THE ASYMPTOTIC SOLUTION.

To characterize large times, we want:

m_∞ = magnetization of configurations in attractor

$C_\infty = \lim_{\tau\to\infty} C(\tau)$ = overlap between configurations visited at $\neq$ times

z_∞ = asymptotic value of Lagrange multiplier

Notice that T.T.I implies: $\partial_\tau C(\tau) \xrightarrow{\tau\to 0,\infty} 0$

Taking $\tau\to\infty$ in equation for $C(\tau)$, we get

$$z_\infty C_\infty + \mu m_\infty^2 - 2\sigma^2 C_\infty^2 / z_\infty = 0 \qquad (z_\infty < 0)$$

We need another equation! In principle, Eq(4).

However, it reads in T.T.I. regime:

$$z_\infty = -2\sigma^2 \int_0^t du\, C^2(u) R(u) - \mu m_\infty^2$$

$u = t-s$

need to know all dynamics, not just asymptotics!

The trick of the potential

SOMPOLINSKY, CRISANTI, SOMMERS 1988

$$\frac{dx(t)}{dt} = z(t)\,x(t) + \mu\, m(t) + \eta(t)$$

multiply by itself:

$$\left(\frac{\partial}{\partial t} - z(t)\right)\left(\frac{\partial}{\partial s} - z(s)\right) x(t)\,x(s) = \mu^2\, m(t)\,m(s) + \eta(t)\,\eta(s)$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - z(t)\right)\left(\frac{\partial}{\partial s} - z(s)\right) C(t,s) = \mu^2\, m(t)\,m(s) + 2\sigma^2 C^2(t,s)$$

$$\xrightarrow{\text{T.T.I.}} \left(\frac{\partial}{\partial \tau} - z_\infty\right)\left(-\frac{\partial}{\partial \tau} - z_\infty\right) C(\tau) = \mu^2\, m_\infty^2 + 2\sigma^2 C^2(\tau)$$

$$\left(-\partial_\tau^2 + z_\infty^2\right) C(\tau) = \mu^2\, m_\infty^2 + 2\sigma^2 C^2(\tau)$$

$$\Rightarrow \partial_\tau^2 C(\tau) = -\frac{\partial}{\partial C} V[C]$$

$$V[C] = \frac{-z_\infty^2 C^2}{2} + \mu^2\, m_\infty^2 C + \frac{2\sigma^2}{3} C^3$$

Hamiltonian dynamics!

Energy conservation: $E[C] = \frac{1}{2}(\partial_z C)^2 + V[C]$

Equating $E[C_0] = E[C_\infty]$ and using $\partial_z C(z) \xrightarrow{z\to\infty,0} 0$:

$V[C_0] = V[C_\infty] \Rightarrow V[C_0] - V[C_\infty] = 0$

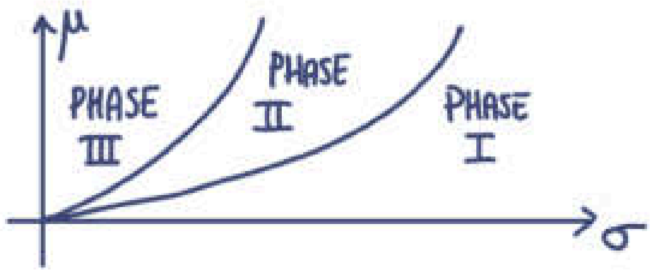
for $z=0$ because $C(z)=C(-z)$. For $z=\infty$, stationarity

This gives third equation:

$$(C_\infty - C_0)^2 \left[4/3\, \sigma^2 C_\infty - \frac{1}{2} z_\infty^2 + \frac{2}{3} \sigma^2 C_0 \right] = 0$$

Compute solutions $\Rightarrow$ dynamical phase diagram!

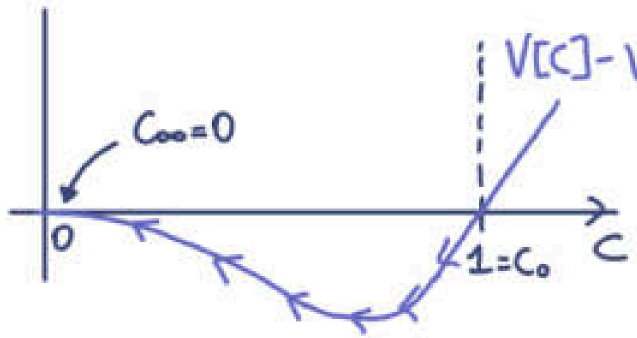
DYNAMICAL PHASE DIAGRAM ($\alpha = 0$)



PHASE I: $m_\infty = 0,\ C_\infty = 0,\ z_\infty = -\frac{2}{\sqrt{3}}\sigma$

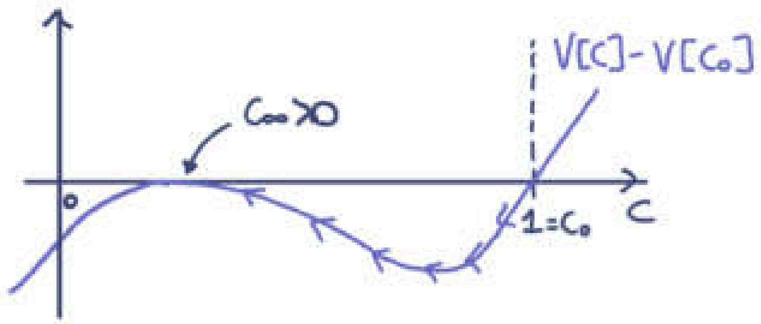
$C_\infty = 0 \Rightarrow$ the system never settles down in T.T.I regime, explores all config. space & decorrelates

The potential:



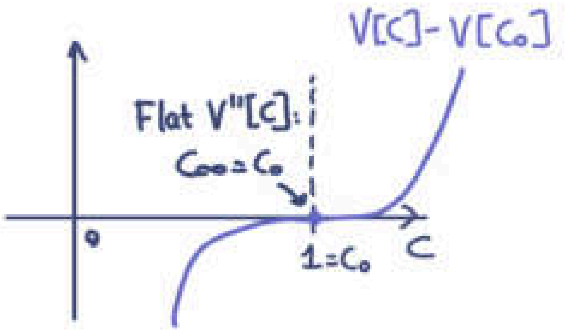
PHASE II: $m_\infty \neq 0,\ 0 < C_\infty = \frac{3\mu^2}{8\sigma^2} - \frac{1}{2} < C_0,\ z_\infty = -\mu$

$0 < C_\infty < 1 \Rightarrow$ The system keeps moving, but in restricted region of config. space.

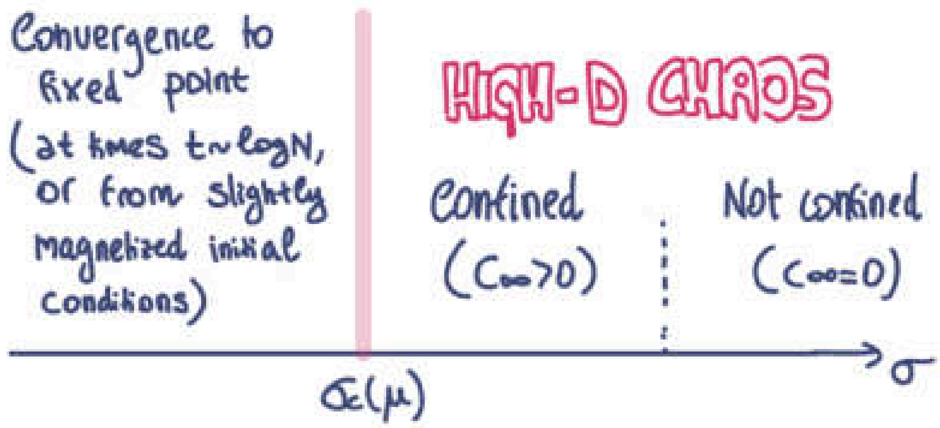


☒ PHASE III: $M_\infty \neq 0$, $C_0 = C_\infty = 1$

The system reaches a fixed point configuration.



Therefore, at fixed $\mu > 0$:



Comments.

1. THE LYAPUNOV. It turns out that when $\mu < \mu_c(\sigma)$, the dynamics is T.T.I, but chaotic. One can compute the Lyapunov exponent explicitly.

   FOURNIER, URBANI, JSTAT 2026
   FOURNIER, URBANI, arXiv: 2601.04702

   Non-linearities turn dynamics into a chaotic one!

2. Generic $\alpha \neq 1$. Same phenomenology.

3. Here, examples of out-of-eq. dynamics in high-D with quenched Random interactions.

   Feedbacks on couplings: Learning, Evolution, Control.

   An example on learning in the chaotic phase (study of learning algorithms with DMFT):

   FOURNIER, URBANI, JSTAT 2023

⇒ Back to our Questions.

Q1. In non-linear case, for $\alpha = 1$ the dynamics exhibits aging, algebraic decay etc.
For $\alpha \neq 1$, trajectories are time-translation invariant but CHAOTIC.

Q2. Again, $\alpha = 1$ is a special point. Can study the "weak-asymmetry" crossover in $\varepsilon = (1-\alpha)$

Q3. For $\alpha = 1$, aging dynamics can be understood from landscape. Lecture notes:

For $\alpha \neq 1$? ⇒ KAC-RICE FORMALISM

For $\alpha = 0$, one can show that in chaotic phase ⇒ many fixed points, all dynamically unstable. Transitions in fixed points distribution mirror those of dynamics

This is what I work on, so if you are interested, we can talk more!